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Adaptive estimation algorithms

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by

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I. INTRODUCTION

In control theory, the objective of the estimation problem is to estimate some signal, using the available noisy measurements. Usually the problem is solved in a statistical context by characterizing the signal and noise as random processes. The estimator is then synthesized according to the statistical models of the signal and noise and some kind of criterion, such as minimum mean-square error.

Wiener (1) made the initial significant contribution to this problem by obtaining the optimal linear, minimum mean-square error estimator for the case where the measurement and signal are scalar, continuous-time, and stationary random processes. His approach has been extended by many others to include nonstationary, discrete-time, and vector random processes. However, for nonstationary and/or vector random processes, his method becomes very complicated and difficult to use.

Kalman (2,3) solved the same problem in an entirely different way. He assumed that the vector signal process could be characterized as the state variables of a linear dynamical system, excited by uncorrelated noise. The measurement process was then assumed to be a linear transformation of the state vector, corrupted by a vector noise process. The resulting Kalman-filter is easily synthesized on a digital computer and readily applicable to nonstationary vector random processes.

Two of the assumptions required for the optimality of

the Kalman-filter are the linearity of the dynamical system and the complete knowledge of its a priori Gaussian statistics. In some real-life situations, these assumptions may not be valid. Cox (4) and many others studied the case where the dynamical system is nonlinear. The situation in which the statistical model is partially unknown provide the motivation for this dissertation. Various approaches (discussed in Chapters II and III) have been used to obtain estimators for this situation. One promising approach is to devise an estimator that will estimate the unknown portion of the statistical model as well as the state variables of the system. This type of estimator will be termed an adaptive estimator since it has the ability to adapt to the initially unknown portion of the statistical model.

In this dissertation, two adaptive estimation algorithms are devised for the case where the signal processes are characterized as state variables of a known first order, linear, vector difference equation with random driving sequences. The measurement process is assumed to be a known linear transformation of the states added to a random noise sequence. In this case, the unknown portion of the statistical model is restricted to be either of the constant covariance matrices of the random driving sequence, or the random noise sequence, or both. Thus, the adaptive estimation algorithms developed in this dissertation estimate the unknown covariance matrices

in addition to estimating the state variables.

The first adaptive estimation algorithm resulted from an intuitive approach to improve the convergence properties of the adaptive estimator developed by Sage and Husa (5). The basic contribution is the idea of reprocessing, using the latest estimates for the unknown covariance matrices to re-estimate the state variables and unknown covariance matrices. Covariance analysis equations are developed to ascertain the convergence properties of the reprocessing estimator for any particular system. In some specific cases, sufficient conditions can be given to ensure that the reprocessing estimator converges to the "true" Kalman-filter (the Kalman-filter using the true values for the unknown covariance matrices).

The second adaptive estimation algorithm resulted from considering the best adaptive estimate to be the mode of the a posteriori probability density of the state trajectory and unknown covariance matrices, conditioned on all the available measurements. The primary contribution is the algorithm used to solve for the conditional mode. Also, sufficient conditions are given to assure convergence of the algorithm to the conditional mode.

The most "optimal" versions of both adaptive estimators feature the reprocessing of all the previous measurements.

In this case, they can only be considered as "off-line" algorithms. However, suboptimal "on-line" algorithms can be realized by reprocessing only a fixed number of the most recent past measurements.

Experimental results comparing the above two adaptive estimators with the estimator of Sage and Husa are shown in Chapter VII.

II. DEFINITION AND MOTIVATION FOR ADAPTIVE ESTIMATORS

The statistical model of the signal and noise process considered in this thesis will first be defined. The Kalman-filter will then be reviewed to show how the minimum mean-square error state estimates can be obtained when the statistical model is completely known. The inadequacies of the Kalman-filter, and of other estimators, in the situation where only part of the statistical model is known, then provide the motivation for discussing the adaptive estimator and defining the estimation problem considered in this thesis.

A. Definition of the Model and Review of Kalman-Filtering

The statistical model of the signal process is assumed to be described by the discrete, linear, vector difference equation[1]

$$x(k) = A(k,k-1)x(k-1) + G(k-1)w(k-1) \qquad (2.1)$$

where $x(k)$ is the n-dimensional state vector at the k^{th} stage, $A(k,k-1)$ is the $n \times n$ state transition matrix, $G(k-1)$ is the $n \times r$ input matrix, and $w(k-1)$ is the r-dimensional

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[1]Unless otherwise specified, all matrices (excluding row or volumn vectors) will be denoted by capital letters, all column vectors will be denoted by lower case letters, and all scalars will be denoted by lower case Greek letters.

random input vector. The statistical model of the measurement process is described by

$$z(k) = H(k)x(k) + v(k) \tag{2.2}$$

where z(k) is the m-dimensional measurement, H(k) is the m x n observation matrix, and v(k) is the m-dimensional random disturbance vector that is corrupting the measurement. The random sequences w(k) and v(k) and the initial state x(0) are all assumed to be Gaussian with[1] (for k = 0,1,...)

$$E[x(0)] = 0, E[w(k)] = 0, E[v(k)] = 0 \tag{2.3}$$

$$E[x(0)w(k)^t] = 0, E[x(0)v(k)^t] = 0, E[w(j)v(k)^t] = 0 \tag{2.4}$$

$$E[w(j)w(k)^t] = Q\delta_{jk} \tag{2.5}$$

$$E[v(j)v(k)^t] = R\delta_{jk} \tag{2.6}$$

where Q is a r x r matrix denoting the covariance matrix of the stationary process noise, w(k), and R is an m x m matrix denoting the stationary measurement noise, v(k). Both Q and R are assumed to be positive definite.

Now an estimate of the state x(k), based upon knowledge of the measurements in Z(j), where

$$Z(j) \triangleq [z(1), z(2), \ldots, z(j)] \tag{2.7}$$

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[1] E[ ] denotes the expectation of (ensemble average of) the quantity within the brackets. D^t denotes the transpose of D. δ_{jk} denotes the Kronecker delta.

will be denoted as $\hat{x}(k/j)$. Specifically, $k > j$ denotes a predicted estimate, $k < j$ denotes a smoothed estimate, and $k = j$ denotes a filtered estimate. If the mean-square error (MSE) is chosen as the optimal criterion, then Kalman (2) shows that the minimizing estimate is given by

$$\hat{x}(k/j) = E[x(k)/Z(j)] \tag{2.8}$$

where $E[x(k)/Z(j)]$ denotes the conditional expectation of $x(k)$, given the knowledge of $Z(j)$.

Now complete knowledge of the statistical model will constitute the knowledge of $A(k,k-1)$, $G(k-1)$, $H(k)$, Q, R, and the structure defined in Equations 2.1 through 2.6. If this is true, then the MSE filtered estimate, $\hat{x}(k/k)$, is given by the Kalman-filter algorithm,

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K(k)\tilde{z}(k/k-1) \tag{2.9}$$

$$\hat{x}(k/k-1) = A(k,k-1)\hat{x}(k-1/k-1) \tag{2.10}$$

$$\tilde{z}(k/k-1) \triangleq z(k) - H(k)\hat{x}(k/k-1) \tag{2.11}$$

where

$$K(k) = P(k/k-1)H(k)^t[H(k)P(k/k-1)H(k)^t + R]^{-1} \tag{2.12}$$

$$P(k/k-1) = A(k,k-1)P(k-1/k-1)A(k,k-1)^t + G(k-1)QG(k-1)^t \tag{2.13}$$

$$P(k/k) = [I - K(k)H(k)]P(k/k-1) \tag{2.14}$$

with the starting values of

$$\hat{x}(0/0) = 0, P(0/0) \triangleq E[x(0)x(0)^t] \quad . \qquad (2.15)$$

A block diagram of the Kalman-filter is shown in Figure 2.1 where

$K(k)$ = the gain matrix for incorporating $z(k)$ into the estimate of $x(k)$

$\tilde{z}(k/k-1)$ = error in predicting (one step ahead) the measurement $z(k)$

$P(k/k-1)$ = covariance of the error in estimating $x(k)$ based on only knowledge of $Z(k-1)$ (a priori covariance matrix)

$P(k/k)$ = covariance of the error in estimating $x(k)$ based on knowledge of $Z(k)$ (a posteriori covariance matrix).

If the Gaussian assumption is dropped, then the Kalman-filter is the minimum MSE linear filter.

From Equations 2.9 through 2.14, and from Figure 2.1, it can be seen that the Kalman-filter is a recursive estimator so that it processes the measurements as they are generated in real time without any growing memory problem. Thus, it is easy to implement on the digital computer for on-line estimation.

B. The Ideal Adaptive Estimator

An important thing to notice about the Kalman-filter is that its structure (the coefficient matrices that describe the algorithm, $A(k,k-1)$, $H(k)$, $G(k-1)$, R, Q, $P(k/k-1)$, $P(k/k)$, $K(k)$), is completely known a priori. In other words, knowing $A(k,k-1)$, $H(k)$, $G(k-1)$, R, Q, and $P(0/0)$ for all k,

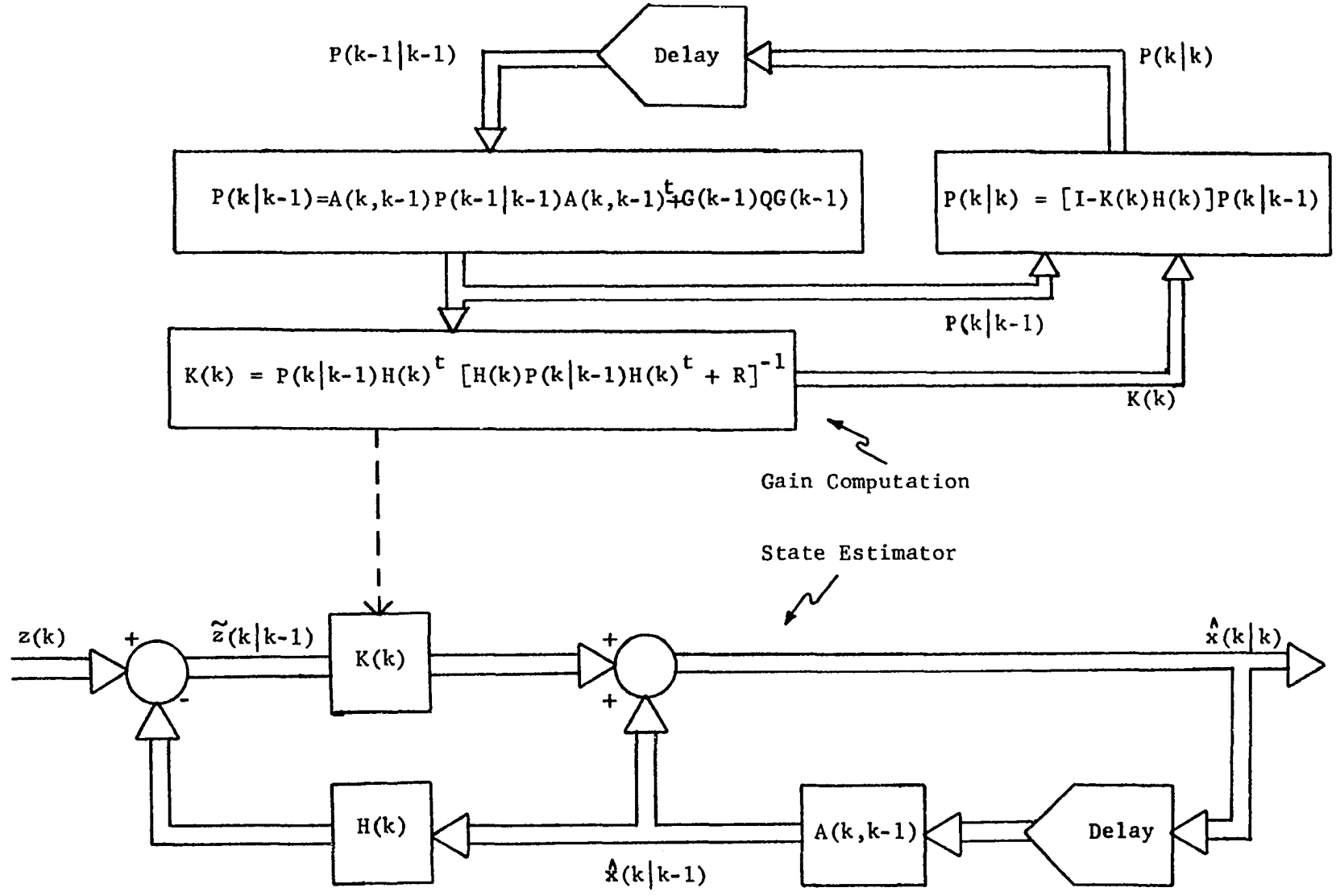

Figure 2.1. Discrete Kalman-filter algorithm

the Kalman gain sequence $\{K(k)\}$ can be generated by Equations 2.12, 2.13, and 2.14 before any measurements are processed. Therefore, this estimator will be termed "non-adaptive" since its structure is independent of the actual sample function measurement sequence.

An obvious disadvantage of the Kalman-filter is that the Gaussian statistical model for the signal and measurement processes must be completely known in order to specify the filter structure. In many cases, however, only part of the statistical model is known (portions of A(k,k-1), H(k), R, Q, G(k-1), or P(0/0) may be unknown). One approach is to try a conservative guess for the unknown portion of the statistical model and use the Kalman-filter with that guess. This results in a suboptimal estimator since the Kalman-filter is not set to the true model. Depending upon the requirements, this may or may not be adequate. Nishimura (6) has developed covariance and sensitivity analysis equations that determine the suboptimality of this approach.

Another approach is to devise an estimator that is insensitive to the unknown portion of the statistical model. Estimators of this type do not need to know anything about the unknown portion of the model. Least squares (7) and minimax (8) estimators are of this type. Again, these estimators are poorer, in the MSE sense, than the true Kalman-filter because they utilize less information about the

statistical model.

Both of the above approaches ignore the possibility of estimating the unknown portion of the statistical model. The ideal adaptive estimator has the capability to estimate the unknown portion of the model and then change its state estimating structure accordingly to improve the state estimates. In other words, the adaptive estimator attempts to extract knowledge of the unknown portion of the model from the available measurement sequence, in addition to estimating the states. An adaptive estimator can then be characterized as a bootstrap procedure, which starts out poorly with only partial knowledge of the model, but gets better and better as more information about the model is abstracted, until it hopefully converges to the Kalman-filter set to the true model.

The advantage of the adaptive estimator is that it should yield better (MSE) estimates of the states than the conservative approaches discussed above. The disadvantage is that the extra estimating capability will probably require a more complex algorithm.

Examples of where an adaptive estimator might be needed are on the high performance aided-inertial navigation systems in ships and aircraft. These systems typically feature an inertial navigator plus one or more other navigation aids, all feeding a Kalman-filter. A common navigation aid for

the ship system is the EM log which measures the speed of the ship relative to the water. When the ship is in a Gulf stream area, the statistical model of the EM log can drastically change, resulting in suboptimal Kalman-filter performance. An adaptive filter could hopefully track this changing EM log model and do a better state estimating job. Similarly, an aircraft system may utilize a doppler radar as an additional navigation aid. The statistics of the doppler radar may change considerably as the terrain under the aircraft changes. An adaptive estimator may, again, be able to track these changes.

C. Definition of the Problem

In this thesis, it is assumed that the only unknown portion of the statistical model is Q and/or R. It will be seen in the next chapter that previous work, applicable to this problem, has resulted in a wide variety of adaptive estimators. Some are, in some sense, optimal or very close to being optimal for small or large numbers of measurements. These estimators are typically very complex and difficult to implement for any practical problem. Others are very simple and easy to implement, but are quite suboptimal. The objective of this thesis is to obtain adaptive estimators that converge better and faster, in the MSE sense, than the simple estimators mentioned above, but which are not prohibitively complex.

III. REVIEW OF PREVIOUS ADAPTIVE ESTIMATORS

Some of the significant previous approaches, applicable to the problem of this thesis, will be reviewed. The resulting adaptive estimators can roughly be categorized into three groups. Those of the first group are optimal, in the MSE sense, for small as well as large amounts of measurement data and their estimators of R and/or Q are consistent. However, these algorithms are typically very complex and difficult to implement. Those of the second group are usable only when a large number of measurements are available. Their estimators of R and/or Q are consistent and may satisfy some type of optimality criterion. Those of the third group are usable for small as well as large measurement applications, but are quite suboptimal to those of the first group. The consistency of their R and Q estimators is questionable but their algorithms are fairly simple to implement. Since the resulting adaptive estimators of this thesis will be compared to the estimator of Sage and Husa (5), a more detailed review will be given of this estimator.

In the first group, the adaptive estimator of Magill (9) assumes the additional restriction that the initially unknown parameters of the statistical model (in this case, the elements of R and/or Q) are characterized by known discrete distributions, each with a finite number of possible values. The optimal (minimum MSE) estimate, $\hat{x}(k/k)$, turns out to be

a weighted sum of "elemental" Kalman-filter estimates, with each "elemental" Kalman-filter set to one of the finite possible values of the unknown parameter vector. The weighting coefficients are obtained via Bayes rule, using the measurement sequence and information from the bank of elemental Kalman-filters. The weighting coefficients provide the adaptive feature since they comprise the portion of the state estimating structure that changes with the measurement sequence. The result is an optimal (MSE) recursive estimator of the states for all time (even when the number of measurements is small). The sufficient condition of ergodicity on the signal and measurement processes assures that Magill's adaptive estimator converge to the Kalman-filter set to the true value of the unknown parameter vector.

Usually the elements of Q and R are considered to have a continuum of possible values (as is assumed in this thesis) so that Magill's approach would need an infinite number of elemental Kalman-filters to realize the MSE adaptive estimate. This being impossible, the suboptimal approach of quantizing the parameter space would result in the complexity and storage requirements of the computer algorithm to increase with the fineness of the quantization and the number of unknown parameters. In realistic problems, this may still prohibit the use of this approach since the number of possible values increases exponentially with the dimensionality of the

parameter space.

Hilborn and Lainiotis (10) extend the above approach to the case of a scalar unknown parameter possessing a continuum of values. However, it requires the elemental estimator to be an explicit nonrecursive function of the unknown parameter, which is quite prohibitive. Sengbush and Lainiotis (11) reduce the number of elemental Kalman-filters needed for the Magill approach by an iterative sequential technique. This scheme is no longer an on-line algorithm but is a possible method of realizing Magill's approach.

An adaptive estimator, belonging to the second group, is one developed by Mehra (12). It uses the basic fact that the suboptimality of the Kalman-filter, set to estimates of R and/or Q, is predictable if the true R and Q are known. Thus by examining the suboptimal performance of a Kalman-filter algorithm, one can estimate R and/or Q. Assuming the system in Equations 2.1 and 2.2 is stationary and that the true or suboptimal Kalman-filter has reached steady state, then asymptotically unbiased estimates of R and/or Q are determined which are shown to be consistent if the system is completely controllable and observable. The R and Q estimators are linear transformations of the estimates of $C(i)$, $i = 0, \ldots N$

$$C(i) \triangleq E[\tilde{z}(k/k-1)\tilde{z}(k-i/k-i-1)^t] \qquad (3.1)$$

denoted as $\hat{C}(i)$, where

$$\hat{C}(i) = \frac{1}{N} \sum_{k=i}^{N} \tilde{z}(k/k-1)\tilde{z}(k-i/k-i-1)^t \qquad (3.2)$$

and where $\tilde{z}(k/k-1)$ are the predicted measurement residuals, defined in Equation 2.11, of a suboptimal Kalman-filter (set to estimates of R and/or Q). The resulting adaptive estimator consists of a Kalman-filter algorithm, set to estimates of R and/or Q, and the estimators of Q, R and C(i), using the sequence $\{z(k/k-1)\}$. The overall algorithm is on-line, but the Kalman-filter algorithm is reset with new estimates R and Q only after every N measurements. N must be fairly large to allow the suboptimal Kalman-filter to be in the steady state for a sufficient amount of time. Thus, this scheme is applicable only to large data situations where quick identification of R and Q is not required. Experimental results show that the R and Q estimators are close to their respective maximum likelihood estimates.

Another estimator of the second group was developed by Kashyap (13). His paper is really concerned with the identification of the unknown statistical model and not with adaptively estimating the state variables. However, his approach can yield estimators for Q and R. Kashyap's approach is to obtain maximum likelihood estimates of the unknown parameters. For Q and R, the likelihood equation is

$$L(R,Q) \triangleq P[Z(N)/R,Q] = P[z(1)/R,Q] \prod_{k=2}^{N} P[z(k)/Z(k-1),Q,R]. \qquad (3.3)$$

Assuming stationarity of the system and that large amounts of data are available (N large), it was shown that consistent estimates of R and Q could be obtained under certain conditions. An on-line approximation (14) to the above approach is also obtainable. However, these methods still suffer from the requirements of a steady state condition and large amounts of data.

The adaptive estimators developed by Shellenbarger (15, 16) belong to the second group and third group. In his first paper (15), he uses an "approximate" maximum likelihood approach by maximizing either P[z(k)/R,Q] or P[z(k)/Z(k-1),R,Q] with respect to Q and/or R at each stage k. The estimators resulting from the first density are very simple on-line algorithms that are independent of the state estimating algorithm (i.e. the Kalman-filter set to the latest estimates of R and/or Q). They are unbiased, and asymptotic stability of the system assures their consistency (17). However, this estimator is very slow to converge and is probably only usable for large data applications, placing it in the second group. The estimators resulting from maximizing the second density are intimately tied in with the Kalman-filter algorithm. The overall adaptive estimator then consists of a Kalman-filter algorithm, using estimates of R and/or Q, and the R and Q estimators, using information from the Kalman-filter. It is recursive, on-line, and fairly

simple to implement. No proof of consistency of the R and Q estimators is available, but it was conjectured that this estimator converged faster than the former one and was usable for small data applications, placing it in the third group.

The second paper of Shellenbarger (16) presents an adaptive estimator of the third group with the R and Q estimates being chosen according to a least squares fit of the elements in $\{\tilde{z}(1/0)\tilde{z}(1/0)^t, \ldots \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t\}$ to their expected values. The resulting adaptive estimator is again intimately tied in with the Kalman-filter algorithm, with the R and Q estimators and the Kalman-filter being mutually dependent on each other. The overall algorithm is recursive and on-line, but somewhat more complicated to implement on the digital computer than the previously discussed estimator. However, Shellenbarger claimed that it is superior. Consistency of the R and Q estimators was not proved.

Smith (18) uses an approximate Bayesian approach to obtain an adaptive estimator when only R is unknown and diagonal. The unknown diagonal elements of R are characterized as random variables possessing inverted-gamma distributions. A proper Bayesian approach would then be to determine the conditional mean of $\mathbf{x}(k)$ from

$$P[x(k)/Z(k)] = \int P[x(k),R/Z(k)]dR \qquad (3.4)$$

where

$$P[x(k),R/Z(k)] = \frac{P[z(k)/x(k),R]P[x(k),R/Z(k-1)]}{P[z(k)/Z(k-1)]} \quad .(3.5)$$

Instead, he assumed that the a priori density $P[x(k),R/Z(k-1)]$ is of the Gaussian-Inverted-Gamma form and forces the a posteriori density $P[x(k),R/Z(k)]$ to be the same form by re-grouping its terms. The Gaussian-Inverted-Gamma characterizing parameters of the a posteriori density are then identified by inspection. Picking an estimator of R such that all the characterizing parameters can be recursively computed, the state estimator becomes the "approximate" conditional mean of the a posteriori density. The resulting overall algorithm consists of a Kalman-filter algorithm tied in with two other recursive equations that propagate the estimate of R and another parameter associated with the Inverted-Gamma density. No proof of consistency is given. The suboptimality of this approach relegates this adaptive estimator to the third group.

Jazwinski (19) develops an adaptive estimator when Q is unknown by examining the predicted measurement residuals N steps ahead. In other words, his Q estimator is a function of the residuals, $\tilde{z}(k+j/k)$, for $j = 1,\ldots,N$ where

$$\tilde{z}(k+j/k) \triangleq z(k+j) - \hat{z}(k+j/k) \qquad (3.6)$$

and

$$\hat{z}(k+j/k) = H(k+j)A(k+j,k)\hat{x}(k/k) \quad . \qquad (3.7)$$

Estimates of Q are obtained by requiring the predicted

residuals to be consistent with their statistical properties, namely

$$\tilde{z}(k+j/k)\tilde{z}(k+j/k)^t = E[\tilde{z}(k+j/k)\tilde{z}(k+j/k)^t] \quad . \tag{3.8}$$

The resulting estimator will have a time lag of N stages since k + N observations must be made before the Q estimate is incorporated into the Kalman-filter algorithm. Under certain conditions the estimate of Q produces the most probable finite sequence of residuals. The overall algorithm is fairly simple to implement for on-line adaptation.

The last adaptive estimator to be discussed is due to Sage and Husa (5). A more detailed review of this algorithm will be given since it will be experimentally compared to the adaptive estimators developed in this thesis. Their starting point is a Bayesian approach in which they attempt to maximize the a posteriori density function,

$$J = P[X(N),Q,R/Z(N)] \tag{3.9}$$

with respect to X(N),Q,R where

$$X(N) \triangleq \{x(0),x(1),\ldots,x(k),\ldots,x(N)\} \quad . \tag{3.10}$$

The elements of Q and R are characterized as random variables possessing uniform a priori distributions. This assumption along with the assumed model described in Chapter II enable the Bayesian equivalent of J to be evaluated as

$$J = C \frac{\exp\{-\frac{1}{2}[\|x(0)\|^2_{P(0/0)^{-1}} + \sum_{k=1}^{N} \|z(k)-H(k)x(k)\|^2_{R^{-1}} + \sum_{k=1}^{N} \|w(k-1)\|^2_{Q^{-1}}]\}}{\prod_{k=1}^{N} |G(k-1)QG(k-1)^t|^{1/2} \; |R|^{N/2}} \tag{3.11}$$

where

$$\|y\|^2_B \triangleq y^t B y \tag{3.12}$$

and

$$|B| \triangleq \text{determinant}(B) \quad . \tag{3.13}$$

The terms in C do not enter into the maximization. The necessary conditions for a maximum of J with respect to X(N), Q, R are in the form of a nonlinear two-point boundary value problem (TPBVP). Their solution of this TPBVP is incorrect (to be discussed in Chapter VI). However, they do not advocate its use because they claim it is too complicated. Instead they advocate a "suboptimal" approach as follows. Their solution to the TPBVP resulted in the R estimator to be given as

$$\hat{R}(k) = \frac{1}{k} \sum_{j=1}^{k} [z(j) - H(j)\hat{x}(j/k)][z(j) - H(j)\hat{x}(j/k)]^t \tag{3.14}$$

where $\hat{x}(j/k)$ is the "smoothed" estimate of x(j), based on k measurements. Since they desired not to use smoothed estimates, their "suboptimal" approach, similar to Jazwinski's, was to determine an estimate of R by requiring

$$\sum_{j=1}^{k} \tilde{z}(j/j-1)\tilde{z}(j/j-1)^t = E\{\sum_{j=1}^{k} \tilde{z}(j/j-1)\tilde{z}(j/j-1)^t\}. \quad (3.15)$$

This resulted in

$$\hat{R}(k) = \frac{1}{k}\sum_{j=1}^{k} \tilde{z}(j/j-1)\tilde{z}(j/j-1)^t - H(j)P(j/j-1)H(j)^t \quad (3.16)$$

which uses only terms generated in the Kalman-filter algorithm. This estimator for R is identical to the one resulting from Shellenbarger's (15) approximate maximum likelihood approach. In recursive form, Equation 3.16 becomes

$$\hat{R}(k) = \frac{1}{k}\{(k-1)\hat{R}(k-1) + \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - H(k)P(k/k-1)H(k)^t\} . \quad (3.17)$$

The same "suboptimal" approach applied to the estimation of Q results in the recursive equation

$$G\hat{Q}(k)G^t = \frac{1}{k}\{(k-1)G\hat{Q}(k)G^t + K(k)\tilde{z}(k/k-1)\tilde{z}(k/k-1)^t K(k)^t + P(k/k) - A(k,k-1)P(k-1/k-1)A(k,k-1)^t\} . \quad (3.18)$$

It is assumed in Sage and Husa's development that G(k) is a constant matrix, i.e. G(k) = G for all k. The state estimating algorithm is the Kalman-filter with the following gain computation equations used instead of Equation 2.12, 2.13, and 2.14.

$$K(k) = P(k/k-1)H(k)^t[H(k)P(k/k-1)H(k)^t + \hat{R}(k-1)]^{-1} \quad (3.19)$$

$$P(k/k-1) = A(k,k-1)P(k-1/k-1)A(k,k-1)^t + G\hat{Q}(k-1)G^t \quad (3.20)$$

$$P(k/k) = [I - K(k)H(k)]P(k/k-1) \quad . \tag{3.21}$$

The overall adaptive estimator of Sage and Husa is shown in Figure 3.1. It is a recursive on-line estimator that has been experimentally shown (5) to be useful in small measurement applications. However, no proof of consistency was given for the R and Q estimators. Consequently, it is relegated to the third group.

In summary, then, the adaptive estimators, discussed in this chapter, were categorized into three groups. The estimators of Magill (9), Hilborn and Lainiotis (10), and Sengbush and Lainiotis (11) belong to the first group. The estimators of Mehra (12), Kashyap (13, 14) and Shellenbarger (15) belong to the second group while the estimators of Shellenbarger (15, 16, 17), Smith (18), Jazwinski (19), and Sage and Husa (5) belong to the third group.

The adaptive estimators of the third group are fairly simple to implement but have questionable optimality and consistency. It is to be remembered that the objective of this thesis is to develop adaptive estimators that are "better" than those of the third group, but are still implementable on the digital computer for on-line as well as off-line estimation.

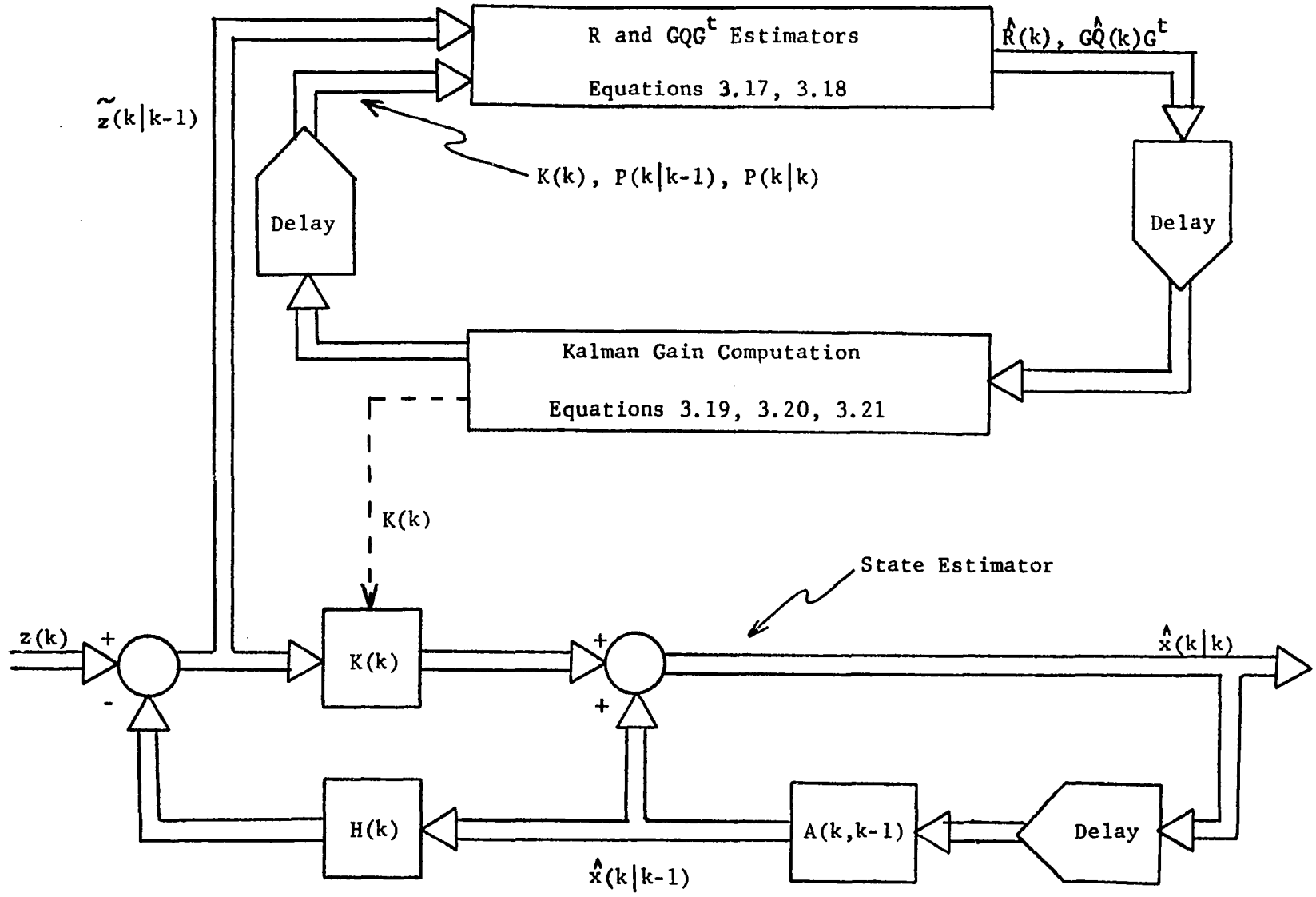

Figure 3.1. Adaptive estimator of Sage and Husa

IV. THE REPROCESSING FILTER

An intuitive approach in obtaining an adaptive estimator, where R and/or Q are unknown, is presented in this chapter. The algorithm, called the reprocessing filter (RF), resulted from an intuitive approach to improve the convergence properties of the adaptive estimator of Sage and Husa (denoted here as SH). The basic idea is that, using the most updated estimates of GQG^t and/or R, reprocessing of all or part of the available measurements through the Kalman-filter would "refine" the estimate of the state trajectory which would in turn yield better estimates of GQG^t and/or R. This idea is more thoroughly discussed in this chapter and the resulting RF algorithm is described for both on-line and off-line applications. Covariance analysis equations are developed to enable an assessment of the convergence properties of the R and GQG^t estimators. In some specific cases, it can be shown that reprocessing produces unbiased and consistent estimates of R and GQG^t.

A. Discussion of the Sage and Husa and Reprocessing Filter Algorithms

The SH estimator, described in the previous chapter, is a recursive algorithm, composed of the Kalman-filter and one or two other recursive equations for estimating R and/or GQG^t. Consequently, it is relatively easy to implement on

a digital computer for on-line or off-line estimation. However, it will be shown, experimentally, that the SH estimator does not converge as rapidly as the RF algorithm and, in some cases, has a tendency to diverge.

The basic idea behind the intuitive approach is that all or part of the past measurements should be reprocessed by the Kalman-filter algorithm after each new estimate at R and/or GQG^t is obtained. This can be best understood by re-examining the SH estimators in Figure 3.1. As each measurement z(k) is processed by the state estimator in the Kalman-filter, new estimates of R and/or GQG^t are calculated by Equations 3.17 and 3.18. These new estimates of R and/or GQG^t are then used in the Kalman gain computation for the processing of the next measurement. Since the Kalman-filter estimates are optimal only when the true values of R and/or GQG^t are used, then the Kalman-filter equations in the SH estimator, using only estimates at R and/or GQG^t, can not yield optimal state estimates. However, these "poor" state estimates, used in Equations 3.17 and 3.18, hopefully yield better estimates of R and/or GQG^t than were previously used. Consequently, the structure within the Kalman-filter equations would slowly become more optimal and the resulting state estimates $\hat{x}(k/k)$ would approach those of the "clairvoyant" Kalman-filter (one with R and/or GQG^t set to their true values).

Now there are two factors that tend to slow down the

convergence of the SH estimator: (1) The equations for estimating R and GQG^t assign equal weighting between the "poor" state estimates of the distant past and the more recent, relatively "better", state estimates. Therefore, many "better" state estimates are needed to swamp out the effects of the "poor" state estimates at the distant past, requiring a longer time for the estimators at R and GQG^t to converge. (2) Even if the Kalman-filter equations could be miraculously set to the true values at R and/or GQG^t at some stage, say k, then the state estimates would still be suboptimal for some time due to the incorrect P(k-1/k-1) and $\hat{x}$(k-1/k-1) that are retained from the previous k-1 stages. These could be considered as incorrect initial values for a Kalman-filter starting at stage k. Nishimura (6) shows that it may take some time before these "initialization" errors propagate out at the Kalman-filter equations. Thus recorrecting the R and/or GQG^t in the Kalman-filter equations of the SH estimator with better estimates will not immediately recorrect for the errors in P(k/k) and $\hat{x}$(k/k). This, again, is a factor which will slow down the convergence of the state estimator.

When, at stage k, new estimates at R and/or GQG^t are computed, the reprocessing of all the measurements z(1),z(2),...z(k) (with the new estimates of R and GQG^t inserted in the Kalman-filter equations throughout the

reprocessing) would generate a set of state re-estimates of $\hat{x}(0/0), \hat{x}(1/1), \ldots, \hat{x}(k/k)$. Now if the new estimates of R and GQG^t are "better" than the ones previously used, the state re-estimates will hopefully be better than the state estimates previously obtained. Similarly, the recomputed value of P(k/k) should be more optimal. Thus, the reprocessing of the measurements, using new estimates of R and GQG^t, will tend to immediately recorrect the values of P(k/k) and $\hat{x}(k/k)$. Also, when Equations 3.17 and 3.18 are used to re-estimate R and/or GQG^t, the "better" state re-estimates would result in "better" re-estimates of R and GQG^t. If desired, this reprocessing scheme could be repeated over and over at stage k for a specified number of times or until successive changes in the re-estimated values of R and/or GQG^t are small.

To help clarify the discussion in this and following chapters, the following notation is defined.

$y_k^j \triangleq$ value of the quantity, y, generated by the Kalman-filter algorithm or the R and/or GQG^t estimators during the j^{th} reprocessing cycle at the k^{th} stage (k measurements available) e.g. $\hat{x}(3/3)_k^j$, $P(3/2)_k^j$, $\hat{R}(5)_k^j$. The symbol, y, without subscripts or superscripts will indicate that it was generated by the SH estimator.

Now as more measurements become available, the re-computed $\hat{x}(k/k)_k^j$, $P(k/k)_k^j$, $\hat{R}(k/k)_k^j$, and/or $G\hat{Q}(k/k)_k^j G^t$ from

the reprocessing scheme can then be used in the regular SH estimator for a number of steps until it is desired to repeat the reprocessing scheme. It is conceivable that one might not desire to reprocess after every measurement because the additional information obtained in the single measurement relative to the information contained in all the past measurements might not justify the extra computer cost of reprocessing.

Actually then, the above intuitive approach attempts to utilize the desirable properties of the SH estimator, namely its simplicity and ease of implementation, while improving its convergence properties by the reprocessing scheme. Obviously, the price that one pays for this improved convergence is that a considerable amount of extra computation is required.

As described so far, the RF algorithm is essentially an off-line algorithm because of the growing memory and growing computation time needed for each successive reprocessing. However, if the reprocessing is carried out only for a fixed number of steps back into the past, then the memory and computation time requirements will be fixed, although considerably more than for the SH estimator. This could then be used for on-line applications. Hopefully, the reprocessing could be performed "far enough" into the past such that most of the advantages of reprocessing are still retained.

B. Description of the Reprocessing Filter Algorithm

The operation of the RF algorithm is depicted in Figure 4.1. N is the stage at which the reprocessing is done, N_B is the number of stages back into the past that the reprocessing is carried, J is the total number of reprocessing cycles at stage N, and M is the number of stages between reprocessing. First, the general operation of the algorithm will be explained. Then particular modes of operation will be pointed out.

The SH estimator is used first to process measurements up to and including the measurement at stage N. At the same time the measurements needed for reprocessing are stored as they become available. At stage N, the reprocessor, shown in Figure 4.2, re-estimates the state trajectory, from stage $N-N_B+1$ to N and simultaneously re-estimates R and/or GQG^t. If $J>1$, then the terminal values from this first cycle of reprocessing (i.e. $\hat{R}(N)_N^1$, $G\hat{Q}(N)_N^1G^t$) are used to re-initialize the reprocessor for another cycle. The operation of the j^{th} cycle of reprocessing at stage N is as follows. The state re-estimation is given by

$$\hat{x}(k/k-1)_N^j = A(k,k-1)\hat{x}(k-1/k-1)_N^j \tag{4.1}$$

$$\tilde{z}(k/k-1)_N^j = z(k) - H(k)\hat{x}(k/k-1)_N^j \tag{4.2}$$

$$\hat{x}(k/k)_N^j = \hat{x}(k/k-1)_N^j + K(k)_N^j\tilde{z}(k/k-1)_n^j$$

$$k = N-N_B+1,\ldots,N \tag{4.3}$$

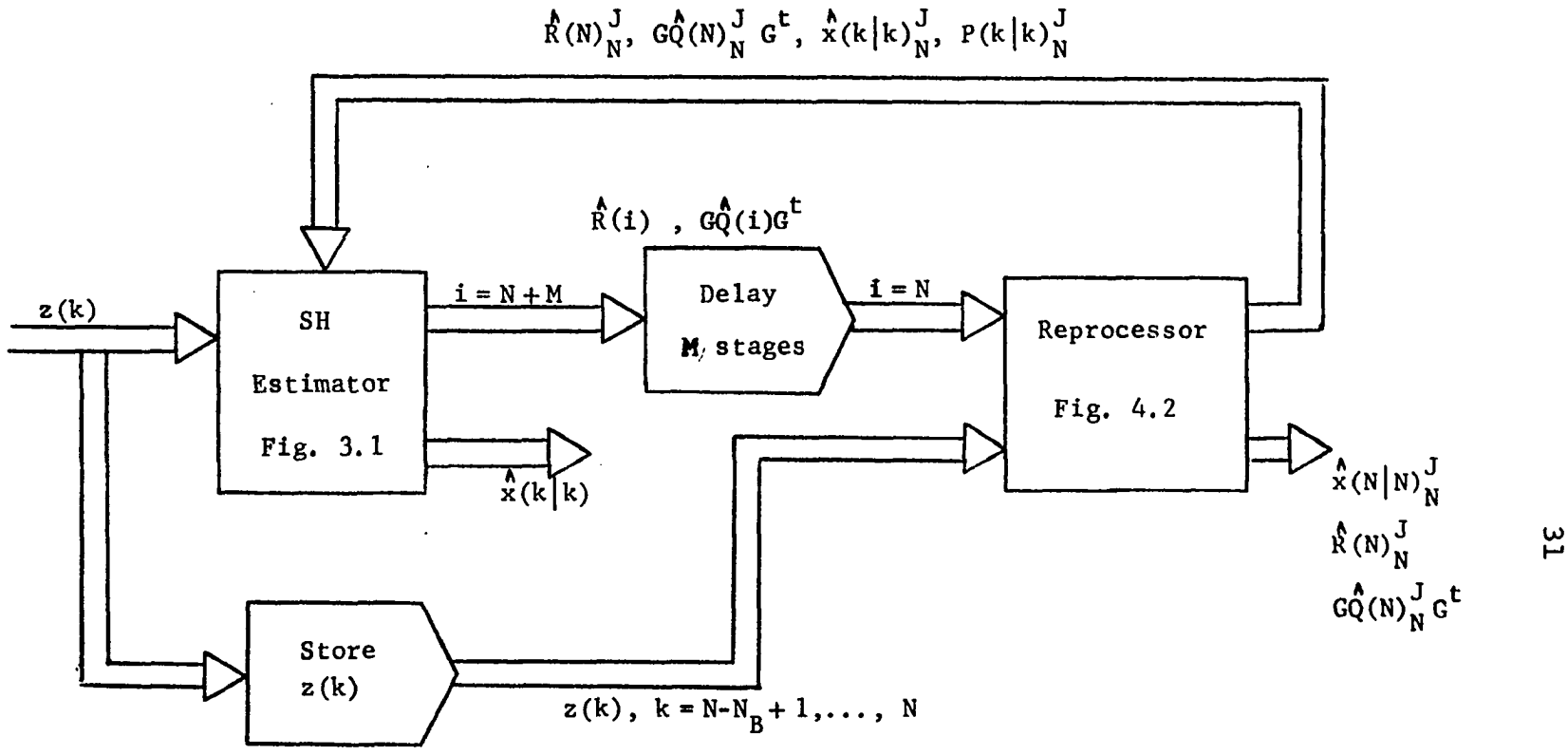

Figure 4.1. Flow diagram of the reprocessing filter

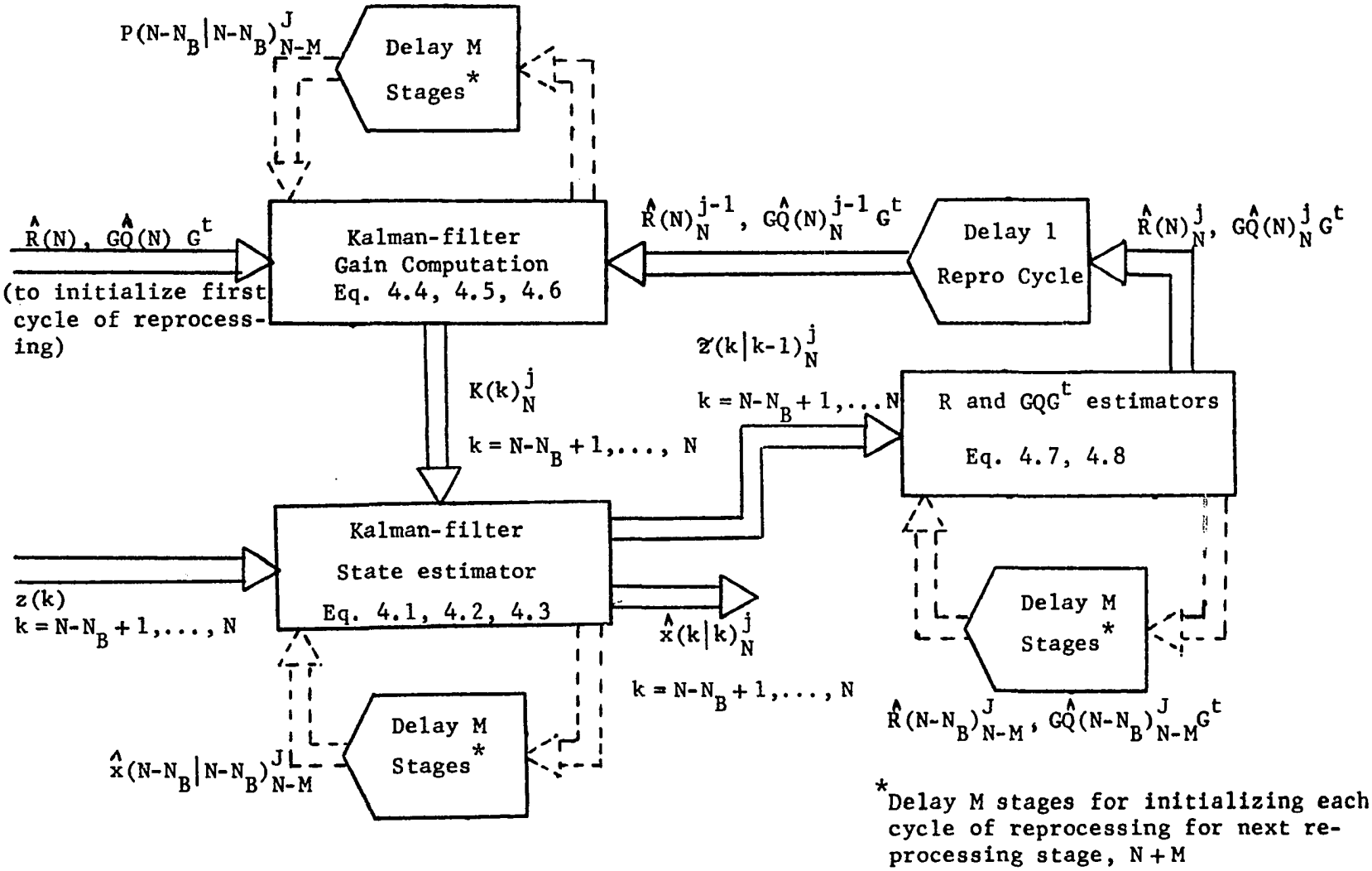

Figure 4.2. Flow diagram of the reprocessor

where the gain computation is

$$P(k/k-1)_N^j = A(k,k-1)P(k-1/k-1)_N^j A(k,k-1)^t + G\hat{Q}(N)_N^{j-1}G^t \qquad (4.4)$$

$$K(k)_N^j = P(k/k-1)_N^j H(k)^t [H(k)P(k/k-1)_N^j H(k)^t + \hat{R}(N)_N^{j-1}]^{-1} \qquad (4.5)$$

$$P(k/k)_N^j = [I - K(k)_N^j H(k)]P(k/k-1)_N^j \qquad (4.6)$$

and the re-estimation of R and/or GQG^t is for $k=N-N_B+1,\ldots,N$,

$$\hat{R}(k)_N^j = \frac{1}{k}\{(k-1)\hat{R}(k-1)_N^j + \tilde{z}(k/k-1)_N^j \tilde{z}(k/k-1)_N^{jt} - H(k)P(k/k-1)_N^j H(k)^t\} \qquad (4.7)$$

$$G\hat{Q}(k)_N^j G^t = \frac{1}{k}\{(k-1)G\hat{Q}(k-1)_N^j G^t + K(k)_N^j \tilde{z}(k/k-1)_N^j \tilde{z}(k/k-1)_N^{jt} K(k)_N^{jt} + P(k/k)_N^j - A(k,k-1)P(k-1/k-1)_N^j A(k,k-1)^t\} \qquad (4.8)$$

where the initial values for the Kalman-filter algorithm of the j^{th} cycle are

$$\hat{x}(N-N_B/N-N_B)_N^j = \hat{x}(N-N_B/N-N_B)_{N-M}^J \qquad (4.9)$$

$$P(N-N_B/N-N_B)_N^j = P(N-N_B/N-N_B)_{N-M}^J \quad . \qquad (4.10)$$

The initial values of the R and GQG^t estimators for the j^{th} cycle are

$$\hat{R}(N-N_B)_N^j = \hat{R}(N-N_B)_{N-M}^J \tag{4.11}$$

$$G\hat{Q}(N-N_B)_N^j G^t = G\hat{Q}(N-N_B)_{N-M}^J \quad . \tag{4.12}$$

The total reprocessing at stage N entails recycling through Equations 4.1-4.12 for j = 1,...,J. The initial values of the estimates of R and GQG^t for the first cycle of reprocessing in the Kalman-filter are the final estimates from the SH estimator, $\hat{R}(N)$ and $G\hat{Q}(N)G^t$, so that in Equations 4.4 and 4.5

$$\hat{R}(N)_N^0 \triangleq \hat{R}(N) \tag{4.13}$$

$$G\hat{Q}(N)_N^0 G^t \triangleq G\hat{Q}(N)G^t \quad . \tag{4.14}$$

At the end of the reprocessing at stage N, the terminal values, $\hat{x}(N/N)_N^J$, $P(N/N)_N^J$, $\hat{R}(N)_N^J$, $G\hat{Q}(N)_N^J G^t$, are used to initialize the SH estimator for processing of the next M measurements, after which the reprocessing is performed again.

It should be noted that the reprocessor is basically composed of the same parts as the SH estimator, except that the re-estimates of R and GQG^t are not reset into the Kalman gain computation after each measurement, but only after each cycle of reprocessing. Also, if the reprocessing only goes back N_B steps, then the proper estimates from the previous reprocessing, at stage N-M, must have been stored to enable the initialization of the reprocessing at stage N. The proper estimates are given in Equations 4.9-4.12.

The particular modes of operation that might be used are as follows. On-line applications would realistically require that the reprocessing be only a fixed number of steps back (N_B fixed) so that computer storage is fixed. One might want to reprocess at every stage ($M = 1$) when only a few measurements are available since each new measurement would add a significant amount of information. However, as the number of measurements become large, then M could be made larger. Usually, because of computer requirements, one might not want to do more than one reprocessing cycle ($J = 1$) at each stage N.

For off-line applications, N could be fixed at the total number of available measurements ($M = 0$) and all of the measurements be reprocessed through a number of cycles ($J > 1$).

C. Convergence of the RF Algorithm

To the extent of the author's knowledge, no work has been done that shows convergence of the SH estimator (or the comparable Shellenbarger estimator of R). This is probably because the resetting of the Kalman-filter equations with new estimates of R and/or GQG^t at each stage prohibitively complicates any accurate analytical analysis. In the case of the RF algorithm, it is possible to develop a computer analysis that will enable one to examine the convergence of the estimator for a particular system without actually performing a Monte Carlo simulation. For the case where the

system is of first order, then it can be shown that the RF estimator will converge under certain conditions.

In the following, it is assumed that all of the measurements will be reprocessed ($N = N_B$) and that only R is unknown. From Equation 4.7 the nonrecursive form for computing $\hat{R}(N)_N^p$ is

$$\hat{R}(N)_N^p = \frac{1}{N} \sum_{k=1}^{N} \tilde{z}(k/k-1)_N^p \tilde{z}(k/k-1)_N^{p\,t} - H(k)P(k/k-1)_N^{p\,t} H(k) \quad . \tag{4.15}$$

Now if the true value of R is used in the Kalman-filter equations, then $P(k/k-1)_N^p$ would be the actual covariance of the state estimation error, $x(k) - \hat{x}(k/k-1)_N^p$. When some other value of R is used, this is no longer true (6). Thus, the matrix computed by the Kalman-filter will be denoted as the computed covariance, $P_c(k/k-1)_N^p$, and the actual covariance of the same estimation error, $x(k) - \hat{x}(k/k-1)_N^p$, will be denoted as $P_a(k/k-1)_N^p$. Therefore, the $P(k/k-1)_N^p$ matrix in Equation 4.15 will be replaced by $P_c(k/k-1)_N^p$ to show that only estimates of R were used in the Kalman-filter algorithm of the reprocessor.

Substituting Equation 2.2 into 4.2, the covariance of $\tilde{z}(k/k-1)_N^p$ is

$$\mathrm{Cov}[\tilde{z}(k/k-1)_N^p] = E\{\tilde{z}(k/k-1)_N^p \tilde{z}(k/k-1)_N^{p\,t}\}$$

$$= E\{[H(k)(x(k) - \hat{x}(k/k-1)_N^p) + v(k)][H(k)(x(k)$$

$$- \hat{x}(k/k-1)_N^p) + v(k)]^t\}$$

$$= H(k)P_a(k/k-1)_N^p H(k)^t + R_t \tag{4.16}$$

where R_t is the true value of R. Thus, the expected value of $\hat{R}(N)_N^p$ becomes

$$E[\hat{R}(N)_N^p] = R_t + \frac{1}{N}\sum_{k=1}^{M} H(k)[P_a(k/k-1)_N^p - P_c(k/k-1)_N^p]H(k)^t . \tag{4.17}$$

Now if the true value of R is used in the reprocessor, then the resulting estimator for R will be unbiased, since $P_a(k/k-1)_N^p$ will equal $P_c(k/k-1)_N^p$. However, in general, the estimator of R will be biased.

To determine how "close" $\hat{R}(N)_N^p$ is to its mean value, the sum of the variances of each element of $\hat{R}(N)_N^p$ is used,

$$d_r(N) \triangleq \sum_{i=1}^{m}\sum_{j=1}^{m} E\{[\hat{R}(N)_N^p - E[\hat{R}(N)_N^p]]_{ij}^2\} \tag{4.18}$$

where $[B]_{ij}$ denotes the ij^{th} element of the matrix B, and d_r is called the "dispersion" in this thesis. From Appendix A,

$$d_r(N) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} Tr[Y_a(i,j)Y_a(i,j)] + \{Tr[Y_a(i,j)]\}^2 \tag{4.19}$$

with

$$Y_a(i,j) =$$

$$\begin{cases} H(j)P_a(j/j-1)_N^p H(j)^t + R_t & i=j \\ \{H(i)P_a(i/i-1)_N^p - R_t[\hat{R}(N)_N^{p-1}]^{-1}H(i)P_c(i/i-1)_N^p\}T(i,j)^t H(j)^t & j>1 \\ H(i)T(j,i)\{P_a(j/j-1)_N^p H(j)^t - P_c(j/j-1)_N^p H(j)^t[\hat{R}(N)_N^{p-1}]^{-1}R_t\} & i>j \end{cases}$$

(4.20)

where

$$T(i,j) \triangleq \prod_{e=1}^{j-i} A(j-e+1,j-e)P_c(j-e/j-e)_N^p[P_c(j-e/j-e-1)_N^p]^{-1} .$$

(4.21)

Now Equations 4.17 and 4.19 will enable a statistical analysis of the convergence of $\hat{R}(N)_N^p$ for a single pass through the reprocessor. Assuming a true and an "estimated" value for R, i.e. some R_t and $\hat{R}(N)_N^{p-1}$, Nishimura (6) shows that $P_a(k/k-1)_N^p - P_c(k/k-1)_N^p$ can be recursively computed. Thus, for some assumed R_t, $E[\hat{R}(N)_N^p]$ and $d_r(N)$ can be computed for many assumed values of $\hat{R}(N)_N^{p-1}$, thus yielding some "feel" as to how the reprocessor will converge, on the average, for different "estimated" values of R.

Now if the system in Equations 2.1 and 2.2 is uniformly completely observable and uniformly completely controllable, then Price (20) shows that $P_a(k/k-1)_N^p$ and $P_c(k/k-1)_N^p$ are

bounded and that the system is uniformly asymptotically stable in the large. It is shown in Appendix B that the above assumption implies that T(i,j) approaches the null matrix as $j >> i$ (similarly when $i >> j$). Using this fact and the boundedness of $P_a(k/k-1)_N^p$ and $P_c(k/k-1)_N^p$, it is further shown in Appendix B that

$$\lim_{N\to\infty} d_r(N) = 0 \tag{4.22}$$

because the terms, $Tr(Y_a(i,j)^2) + [Tr(Y_a(i,j))]^2$, become insignificant as i and j get very far apart, and the remaining terms do not increase in magnitude indefinitely as N gets large.

Thus, under these conditions, $\hat{R}(N)_N^p$ converges in a mean-square sense to its average as N becomes large. Also, if the true value of R is used in the reprocessor, then $\hat{R}(N)_N^p$ converges to the true value of R.

Consequently, in general, it can be seen that for one cycle through the reprocessor, $\hat{R}(N)_N^p$ will not converge to R_t for large N. The question then is, "Does successive reprocessing as N becomes large cause the successive re-estimates of R to converge to R_t?" In other words, for N large, does

$$\lim_{p\to\infty} \hat{R}(N)_N^p = R_t \quad ?$$

If the assumptions of observability and controllability

hold, then by the previous development, for N large,

$$\hat{R}(N)_N^p \approx R_t + \frac{1}{N} \sum_{k=1}^{N} H(k)[P_a(k/k-1)_N^p - P_c(k/k-1)_N^p]H(k)^t \quad . \tag{4.23}$$

Using the recursive equations of Nishimura (6) for computing $P_a(k/k-1)_N^p - P_c(k/k-1)_N^p$ for an assumed true R, R_t, and an assumed estimate of R, $\hat{R}(N)_N^{p-1}$, Equation 4.23 can be used to calculate $\hat{R}(N)_N^p$, the re-estimated R. A comparison of $\hat{R}(N)_N^p$ and $\hat{R}(N)_N^{p-1}$ will then show whether $\hat{R}(N)_N^p$ is "closer" to R_t than $\hat{R}(N)_N^{p-1}$. This could then be repeated as many times as desired to see whether successive reprocessing will yield a converging estimate of R.

For the case where the system in Equations 2.1 and 2.2 is a 1st order stationary system, it is shown in Appendix B that

$$|\hat{R}(N)_N^p - R_t| < |\hat{R}(N)_N^{p-1} - R_t| \tag{4.24}$$

so that for this case, one can be assured that successive reprocessing will converge.

For the case where Q is unknown or both R and Q are unknown, the analysis proceeds similarly. It is shown in Appendix C that the estimators of GQG^t and R are generally unbiased but will all converge to their means in a mean-square sense for a large number of measurements under the

same sufficient conditions mentioned previously. The estimator of GQG^t, when R is known, can also be shown to converge to the true value, GQ_tG^t, by successive reprocessing for the scalar stationary case. In all cases, however, equations are developed which will enable a computer analysis of the convergence of the R and GQG^t estimators for any type of system.

V. A BAYESIAN APPROACH - THE A POSTERIORI CONDITIONAL MODE

A Bayesian approach to the problem of this thesis is considered in this chapter. The best adaptive estimate of the state is chosen to be the mode of the a posteriori probability density function of the state trajectory, R, and/or Q, conditioned by all the available measurements. This will be referred to as the maximum a posteriori estimate or MAP estimate. Various motivations are discussed for choosing this as the "best" estimate. The a posteriori density is evaluated for the assumed system model in Equations 2.1 through 2.6. Finally two convenient a priori characterizations of R and Q, the multivariate uniform distribution and the inverted Wishart distribution, are discussed. The algorithm for finding the a posteriori conditional mode is developed in Chapter VI.

A. Definition and Motivation for the MAP Estimate

The basic idea behind the Bayesian approach is that the estimates are extracted from the a posteriori conditional density, the probability density function of the unknown random variables, conditioned by the available measurements. Here, the unknowns are characterized as random variables with some known or assigned a priori distribution. Bayes rule is then used to compute the a posteriori density

function. An attractive feature of this approach is that it permits a priori knowledge of the unknown variables to be reflected in the choice of their a priori distributions.

The "best" adaptive estimate of the state x(k) is defined to be $\hat{x}(k/k)$, where $\hat{x}(k/k)$ is the terminal member of the sequence x(0),x(1),...,x(k) that, along with Q and R, jointly maximize the a posteriori probability density function,

$$P[X(k),Q,R/Z(k)] \quad . \tag{5.1}$$

Estimates of X(k), Q, and R will be called MAP estimates. (Note that this is the same starting point as in Sage and Husa (5). However, the resulting algorithm in this thesis will be considerably different.) Some of the reasons for choosing this definition are as follows:

(1) It considers the problems of unknown states and unknown R and/or Q as a unified problem rather than two separate problems as in Shellenbarger's estimators (15, 17). In that case the state estimator and the estimators for R and Q are developed separately, each assuming that the other is doing a perfect job. The same can be said for the final "suboptimal" estimators advocated by Sage and Husa (5).

(2) It makes sense, statistically, to pick the conditional mode in that the most probable (likely)

values of X(k), Q, R are chosen, given the information of the available measurements and a priori characterizations.

(3) If Q and R are known, then it can be shown (4) that the MAP estimate yields the Kalman-filter which is the best MSE estimator for the assumed model. Therefore, if the R and Q estimating structure is converging to their true values, then the state estimating structure will converge to the optimal Kalman-filter.

(4) Finally, the MAP estimator offers a workable solution (i.e., it yields a computable answer).

B. Evaluation of the A Posteriori Probability Density

Using Bayes rule, the a posteriori density in Equation 5.1 can be rewritten as

$$P[X(k),Q,R/Z(k)] = \frac{P[Z(k)/X(k),Q,R]P[X(k)/Q,R]P[Q,R]}{P[Z(k)]} \quad . \tag{5.2}$$

Since it is desired to maximize the above density with respect to (X(k)Q,R), then it is sufficient to maximize the numerator since P[Z(k)] is not an explicit function of (X(k),Q,R). Therefore, let the numerator be J or

$$J = P[Z(k)/X(k),Q,R]P[X(k)/Q,R]P[Q,R] \quad . \tag{5.3}$$

It will be assumed, here, that all of Q and R are

unknown; that r = n; and that G(k) is nonsingular. The relaxation of these assumptions will be discussed later. The independence of the measurement noise sequence, $\{v(k)\}$, enables

$$P[Z(k)/X(k),Q,R] = \prod_{j=1}^{k} P[z(j)/x(j),R] \tag{5.4}$$

where Q does not influence the measurement if X(k) is given. The independence of the system driving sequence $\{w(k)\}$ results in x(k), given by Equation 2.1, being a Markov sequence (21), which allows

$$P[X(k)/Q,R] = P[x(0)] \prod_{j=1}^{k} P[x(j)/x(j-1),Q] \tag{5.5}$$

since R does not influence the propagation of the state in Equation 2.1. Now from Chapter II, x(0) is Gaussian (Normal) with $E[x(0)] = 0$ and $Cov[x(0),x(0)] = P(0/0)$ which can be written as

$$x(0) \sim N[0,P(0/0)] \tag{5.6}$$

where Equation 5.6 is based on a priori knowledge of the initial state. Since v(k) is Gaussian, then by Equation 2.2,

$$(z(j)/x(j),R) \sim N[H(j)x(j),R] \quad . \tag{5.7}$$

Also, w(j-1) is Gaussian, so from Equation 2.1,

$$(x(j)/x(j-1),Q) \sim N[A(j,j-1)x(j-1),G(j-1)QG(j-1)^{t}]. \tag{5.8}$$

Since G(j) is invertible and Q is positive definite, then

$G(j-1)QG(j-1)^t$ is nonsingular. Using Equations 5.6, 5.7, 5.8 in Equations 5.4 and 5.5, and then substituting the results in Equation 5.3,

$$J = \frac{\left\{\exp -.5\left[\|x(0)\|^2 P(0/0)^{-1} + \sum_{j=1}^{k} \|z(j) - H(j)x(j)\|^2 R^{-1} + \sum_{j=1}^{k} \|x(j) - A(j,j-1)x(j-1)\|^2 [G(j-1)QG(j-1)^t]^{-1}\right]\right\} P[Q,R]}{(2\pi)^{\frac{km+(m+1)n}{2}} |M|^{\frac{1}{2}} |R|^{\frac{k}{2}} \prod_{j=1}^{k} |G(j-1)QG(j-1)^t|^{\frac{1}{2}}} .$$

(5.9)

Now Equation 2.2 yields,

$$G(j-1)w(j-1) = x(j) - A(j,j-1)x(j-1) \tag{5.10}$$

so that since G(j) is invertible,

$$\|x(j) - A(j,j-1)x(j-1)\|^2 [G(j-1)QG(j-1)^t]^{-1} = \|w(j-1)\|^2 Q^{-1} .$$

(5.11)

Therefore, J becomes J^* where

$$J^* = \frac{\left\{\exp -.5\left[\|x(0)\|^2 P(0/0)^{-1} + \sum_{j=1}^{k} \|z(j) - H(j)x(j)\|^2 R^{-1} + \sum_{j=1}^{k} \|w(j-1)\|^2 Q^{-1}\right]\right\} P[Q,R]}{C|R|^{\frac{k}{2}} \prod_{j=1}^{k} |G(j-1)QG(j-1)^t|^{\frac{1}{2}}}$$

(5.12)

where C contains terms that do not enter into the maximization. From Equations 5.10 and 5.11, maximizing J with

respect to (X(k),Q,R) is equivalent to maximizing J* with respect to (X(k),Q,R,W(k-1)), subject to the equality constraints,

$$x(j) = A(j,j-1)x(j-1) + G(j-1)w(j-1) \quad j = 1,\ldots,k \qquad (5.13)$$

where

$$W(k) \triangleq \{w(0),w(1),\ldots,w(k)\} \quad . \qquad (5.14)$$

If the assumption of $n = r$ and G(k) being invertible is relaxed, namely, $n > r$ and G(k) is of rank r, then an expression almost identical to Equation 5.12 can be obtained. The only difference would be that $|G(j-1)QG(j-1)^t|$ would be replaced by $|G_1(j-1)QG_1(j-1)^t|$ where $G_1(j)$ is an $r \times r$ invertible submatrix of G(j). Since the procedure is the same, this case will not be pursued any further.

C. A Priori Characterization of Q and R

In order to completely specify J*, the density P[Q,R] remains to be specified. Two different distributions will be used to characterize the elements of Q and R. First of all, it will be assumed that the elements of Q are independent of the elements of R so that

$$P[Q,R] = P[Q]P[R] \quad . \qquad (5.15)$$

Thus, the characterization of each matrix can be specified independently.

Now it may be desirable to specify the random variables

of R to belong to a multivariate uniform distribution with the density function

$$P[R] = \begin{cases} C_R & ; \quad R \in E_R^* \\ 0 & \text{otherwise} \end{cases} \qquad (5.16)$$

where C_R is a normalizing constant so that P[R] is a density and E_R^* is a specified region in the $\frac{m(m-1)}{2}$ Euclidean space of the $\frac{m(m-1)}{2}$ elements of R. E_R^* is chosen according to the a priori knowledge of the user. An example is shown in Appendix D. This characterization might be desirable for the case where very little is known about R except that the elements are within some region, E_R^*. The same characterization can also be given to Q in an equation similar to Equation 5.16.

Another possible choice for characterizing Q and R is related to the Wishart distribution (22). In the field of multivariate statistical analysis, this distribution is often used to characterize the inverse of unknown covariance matrices for Bayesian estimation problems (22). This distribution can be thought of as a multivariate version of the Chi-square distribution or Gamma distribution. So it seems appropriate in this case to characterize R^{-1} and Q^{-1} by the Wishart densities $P[R^{-1}]$ and $P[Q^{-1}]$. To be consistent with the rest of this thesis, the corresponding densities for R and Q are derived in Appendix D. The density of R, designated

as the "inverted Wishart" density is

$$P[R] = \begin{cases} \dfrac{\exp\{-\mathrm{Tr}(B_R R^{-1})\}}{K_R |R|^{\lambda_R + m}} & \text{for R positive definite} \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

where B_R is a known, positive definite, and symmetric matrix. The parameter, λ_R, is a known positive scalar constant that indicates the amount of concentration about the mode,

$$\frac{B_R}{\lambda_R + m}$$

of P[R]. Also, K_R is a known scalar normalizing constant. If R is known to be a scalar, then the characterization of R becomes an inverted-Gamma density function. A graph of this density is shown in Appendix D. The same characterization can also be given to Q in an expression similar to Equation 5.17.

The characterization of R and Q then completes the evaluation of the a posteriori density.

VI. ALGORITHM FOR THE A POSTERIORI CONDITIONAL MODE

Necessary conditions for a maximum of the a posteriori density are determined for both the uniform and inverted Wishart characterizations of Q and R. The resulting necessary conditions which constitute a nonlinear TPBVP are then approximately solved by a reprocessing algorithm, called the MAP estimator. It is composed of the Kalman-filter algorithm, a fixed-interval smoothing algorithm, and the estimators for R and Q. A discussion is also given to differentiate between the solution of Sage and Husa (5) and the MAP estimator.

As was shown in Chapter V, it was desired to maximize J^*, given by Equation 5.12, with respect to $\{X(k), Q, R, W(k-1)\}$ subject to the equality constraint in Equation 5.13. Since $\ln CJ^*$ is a monotonically increasing function of J^*, then maximizing J^* would yield the same result as maximizing $\ln CJ^*$ or minimizing $-\ln CJ^*$. Equivalently, it is then desired to minimize, from Equations 5.12 and 5.15,

$$I \triangleq -\ln CJ^* = \frac{k}{2}\ln|R| + \frac{1}{2}\sum_{j=1}^{k} \ln|G(j-1)QG(j-1)^t|$$

$$+ \frac{1}{2}[\|x(0)\|^2_{P(0/0)^{-1}} + \sum_{j=1}^{k} \|z(j) - H(j)x(j)\|^2_{R^{-1}}$$

$$+ \sum_{j=1}^{k} \|w(j-1)\|^2_{Q^{-1}}] - \ln P[Q] - \ln P[R] \qquad (6.1)$$

with respect to $\{X(k),Q,R,W(k-1)\}$, subject to the constraint

$$x(j) = A(j,j-1)x(j-1) + G(j-1)w(j-1) \quad j = 1,\ldots,k \quad . \tag{6.2}$$

A. Necessary Conditions - Uniform Characterization of R and Q

First of all it will be assumed that the unknown R and Q are characterized by the multivariate uniform density, described in Chapter V. Therefore, let

$$P[R] = \begin{cases} C_R & R \in E_R^* \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

$$P[Q] = \begin{cases} C_Q & Q \in E_Q^* \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

Therefore,

$$-\ln P[R] = \begin{cases} -\ln C_R & R \in E_R^* \\ \infty & \text{otherwise} \end{cases} \tag{6.5}$$

and

$$-\ln P[Q] = \begin{cases} -\ln C_Q & Q \in E_Q^* \\ \infty & \text{otherwise} \end{cases} \quad . \tag{6.6}$$

Now $-\ln P[R]$ and $-\ln P[Q]$ may be thought of as "penalty" functions which give an infinite penalty (infinite value of I) if any of the constraints ($R \varepsilon E_R^*$, $Q \varepsilon E_Q^*$) are violated. Hence, minimization of I is the same as minimizing

$$I^* = I + \ln P[Q] + \ln P[R]$$

$$= \frac{k}{2}\ln|R| + \frac{1}{2}\sum_{j=1}^{k} \ln|G(j-1)QG(j-1)^t|$$

$$+ \frac{1}{2}[||x(0)||^2 P(0/0)^{-1} + \sum_{j=1}^{k} ||z(j) - H(j)x(j)||^2 R^{-1}$$

$$+ \sum_{j=1}^{k} ||w(j-1)||^2 Q^{-1}] \qquad (6.7)$$

subject to the additional inequality constraints

$$R \varepsilon E_R^* \qquad (6.8)$$

$$Q \varepsilon E_Q^* \quad . \qquad (6.9)$$

Now a possible procedure to use would be to first find the minimum of I^*, neglecting the inequality constraints in Equations 6.8 and 6.9. Then if this "unconstrained" minimum violates any of the inequality constraints, the violating element would be reset to its nearest boundary value. It is well known (23) that this is not always an optimal procedure in that the "constrained" minimum may be some other value. However, this resetting method will be used

since it is a simple and easy way to handle the inequality constraints.

Consequently, the inequality constraints in Equations 6.8 and 6.9 will be initially disregarded in finding the minimum of I* in Equation 6.7. After an estimate of R and/or Q is obtained, they will then be checked to see if any are violated.

Thus, the immediate problem at hand is to find necessary conditions for a minimum of I* in Equation 6.7 with respect to $\{X(k),Q,R,W(k-1)\}$ subject to the equality constraints of Equation 6.2. Using Lagrange multipliers, s(j), the equality constraints are adjoined to the cost function I* to obtain

$$I^{**} = I^{*} + \sum_{j=1}^{k} s(j-1)^{t}[x(j) - A(j,j-1)x(j-1) - G(j-1)w(j-1)] \tag{6.10}$$

yielding the equivalent minimization problem of I** with respect to $\{X(k),Q,R,W(k-1),S(k-1)\}$ where

$$S(k-1) \triangleq \{s(0),s(1),\ldots,s(k-1)\} \quad . \tag{6.11}$$

In Appendix E, the partial derivatives of I** with respect to $\{X(k),Q,R,W(K-1),S(k-1)\}$ are set to zero to obtain the necessary conditions for a local extremum of I**. Now the solution to these necessary conditions will all be based on the observed measurement sequence in Z(k). Therefore, these

solutions will be indicated as $\hat{x}(0/k)$, $\hat{x}(1/k)$, ..., $\hat{x}(k/k)$, $\hat{R}(k)$, $\hat{Q}(k)$, $\hat{w}(0/k)$, $\hat{w}(1/k)$, ..., $\hat{w}(k-1/k)$, $s(0/k)$, $s(1/k)$, ..., $s(k-1/k)$ where $\hat{x}(j/k)$ is the value of $x(j)$ that jointly extremizes the cost function I**, based on k measurements. A similar definition is given to the rest of the above variables. From Appendix E, the resulting necessary conditions become,

$$\hat{x}(0/k) = P(0/0)A(1,0)^t s(0/k) \tag{6.12}$$

$$s(j-1/k) = A(j+1,j)^t s(j/k) + H(j)^t \hat{R}(k)^{-1}[z(j)-H(j)\hat{x}(j/k)]$$

$$j = 1,\ldots,k \tag{6.13}$$

$$s(k/k) = 0 \tag{6.14}$$

$$\hat{x}(j+1/k) = A(j+1,j)\hat{x}(j/k) + G(j)\hat{w}(j/k) \quad j = 0,\ldots,k-1 \tag{6.15}$$

$$\hat{w}(j/k) = \hat{Q}(k)G(j)^t s(j/k) \quad j = 0,\ldots,k-1 \tag{6.16}$$

$$\hat{R}(k) = \frac{1}{k}\sum_{j=1}^{k}[z(j) - H(j)\hat{x}(j/k)][z(j) - H(j)\hat{x}(j/k)]^t \tag{6.17}$$

$$\hat{Q}(k) = \frac{1}{k}\sum_{j=1}^{k}\hat{w}(j-1/k)\hat{w}(j-1/k)^t \quad . \tag{6.18}$$

The approximate solution of these equations is given in section C of this chapter.

For the case where only portions of R or Q are unknown, the a posteriori density has the same evaluation as was shown

in Chapter V with the exception that the density functions of Q and R characterize only the unknown portions. Thus, the resulting equivalent minimization problem is the same as the above except that only the partial derivatives of I** with respect to the unknown portions of Q and R are set to zero. The resulting necessary conditions are similar to those of Equations 6.12 through 6.18 and will not be pursued any further in this thesis.

B. Necessary Conditions - Inverted Wishart Characterization of R and Q

Now it will be assumed that the unknown Q and R are characterized by

$$P[R] = \begin{cases} \dfrac{\exp\{-\mathrm{Tr}(B_R R^{-1})\}}{K_R |R|^{\lambda_R + m}} & R \text{ is p.d.}^1 \\ 0 & \text{otherwise} \end{cases} \tag{6.19}$$

$$P[Q] = \begin{cases} \dfrac{\exp\{-\mathrm{Tr}(B_Q Q^{-1})\}}{K_Q |Q|^{\lambda_Q + r}} & Q \text{ is p.d.} \\ 0 & \text{otherwise} \end{cases} \tag{6.20}$$

where B_R, B_Q, λ_R, λ_Q, K_R, K_Q are defined in Chapter V. From Equations 6.19 and 6.20 (neglecting the constants K_R and K_Q which do not affect the minimization),

[1] p.d. stands for positive definite.

$$-\ln P[R] - \ln P[Q] = \left\{ \begin{array}{ll} (\lambda_R+m)\ln|R| + \mathrm{Tr}(B_R R^{-1}) & R \text{ is p.d.} \\ \infty & \text{otherwise} \end{array} \right\} + \left\{ \begin{array}{ll} (\lambda_Q+r)\ln|Q| + \mathrm{Tr}(B_Q Q^{-1}) & Q \text{ is p.d.} \\ \infty & \text{otherwise} \end{array} \right\} . \tag{6.21}$$

Therefore, using Equation 6.21 in Equation 6.1, I must be minimized with respect to $\{X(k),R,Q,W(k-1)\}$ subject to the equality constraints of Equation 6.2. However, from Equation 6.21, this is equivalent to minimizing

$$\begin{aligned} I^* = {} & \frac{k}{2}\ln|R| + \frac{1}{2}\sum_{j=1}^{k} \ln|G(j-1)QG(j-1)^t| \\ & + [\,\|x(0)\|^2_{P(0/0)^{-1}} + \sum_{j=1}^{k} \|z(j) - H(j)x(j)\|^2_{R^{-1}} \\ & + \sum_{j=1}^{k} \|w(j-1)\|^2_{Q^{-1}}] + (\lambda_R+m)\ln|R| + \mathrm{Tr}(B_R R^{-1}) \\ & + (\lambda_Q+r)\ln|Q| + \mathrm{Tr}(B_Q Q^{-1}) \end{aligned} \tag{6.22}$$

subject to the additional inequality constraints

$$R \text{ is p.d., } Q \text{ is p.d.} \tag{6.23}$$

It turns out that the necessary conditions for extremizing I* will always yield a p.d. estimate for R and Q. There-

fore, the inequality constraints can be neglected. The objective then is to minimize I* with respect to $\{X(k),R,Q,W(k-1)\}$ subject to Equation 6.2.

From Appendix E, Lagrange multiplier vectors are used to attach the equality constraints to I*. The partial derivatives of the resultant cost functions are then set to zero to obtain the necessary conditions for a local extremum.

The results are the same as in Section A except that Equations 6.17 and 6.18 must be replaced by

$$\hat{R}(k) = \frac{1}{k+2(\lambda_R+m)} \left\{ \sum_{j=1}^{k} [z(j) - H(j)\hat{x}(j/k)][z(j) - H(j)\hat{x}(j/k)]^t + 2B_R \right\} \tag{6.24}$$

$$\hat{Q}(k) = \frac{1}{k+2(\lambda_Q+r)} \left\{ \sum_{j=1}^{k} \hat{w}(j-1/k)\hat{w}(j-1/k)^t + 2B_Q \right\} . \tag{6.25}$$

Note that since B_Q and B_R are positive definite, then $\hat{R}(k)$ and $\hat{Q}(k)$ will always be positive definite.

Before proceeding on, it will be pointed out that necessary conditions for an extremum of

$$P[X(k)/Z(k)] = \int\int P[X(k),Q,R,/Z(k)]dQdR \tag{6.26}$$

are developed in Appendix F. Since the integration can be performed, the resulting maximization is only with respect to $\{X(k)\}$. However, the resulting necessary conditions can be shown to be almost identical to those

given in this chapter, except that the estimators for R and Q are slightly different when the number of measurements are small.

C. Approximate Solution of the Necessary Conditions

With either of the characterizations assumed for Q and R, the resulting set of necessary conditions form a nonlinear TPBVP, where Equations 6.12 and 6.14 constitute the two "split" boundary conditions. To get the local extremum solution, one must solve the equations (Equations 6.12 through 6.16 plus Equation 6.17 or 6.24 and Equations 6.18 or 6.25) simultaneously. Substituting Equations 6.17 and 6.18 into Equations 6.13 and 6.16, one could try to solve for $\hat{x}(i/k)$'s. It can be seen that this is practically impossible to do directly since they will be nonlinear functions of the measurements. However, if $\hat{Q}(k)$ and $\hat{R}(k)$ are assumed to be given, then Equations 6.12 through 6.16 constitute a linear TPBVP, which can be easily solved by the Kalman-filter algorithm coupled with the fixed-interval smoothing algorithm. This derivation will now be described.

Assuming that $\hat{R}(k)$ and $\hat{Q}(k)$ are fixed in Equations 6.12 through 6.16, Cox (4) shows that the Kalman-filter can be derived by successively solving each linear TPBVP ending at stages $j = 0,1,2,\ldots,k$. The estimate, $\hat{x}(j/j)$, is then the terminal member of the sequence, $\{\hat{x}(i/j)\}$, $i = 0,1,\ldots,j$ for the linear TPBVP with j measurements. It turns out that

$\hat{x}(j/j)$ can be linearly and simply related to $\hat{x}(j-1/j-1)$ if the values of $\hat{R}(k)$ and $\hat{Q}(k)$ are assumed fixed with respect to each successive TPBVP, up to and including stage k. This relationship between successive terminal members of successive TPBVP is the Kalman-filter algorithm. Now, once $\hat{x}(k/k)$ has been recursively computed, the sequence $\{\hat{x}(i/k)\}$, $i = 0,1,\ldots,$ k-1 can be computed "backwards", using the fixed interval-smoothing algorithm of

$$\hat{x}(j/k) = \hat{x}(j/j) + P(j/j)A(j+1,j)^{t}s(j/k) \tag{6.27}$$

$$s(j-1/k) = A(j+1,j)^{t}s(j/k) + H(j)^{t}\hat{R}(k)^{-1}[z(j)-H(j)\hat{x}(j/k)] \tag{6.28}$$

with the boundary condition

$$s(k/k) = 0 \quad . \tag{6.29}$$

The procedure is to first precompute and store $\hat{x}(j/j)$ and $P(j/j)$ for $j = 0,\ldots,k$ by the Kalman-filter algorithm, in addition to storing $z(1),\ldots,z(k)$. The smoothing equation is then initiated at $j = k$ with $\hat{x}(k/k)$ and $s(k/k)$ in Equation 6.28. The smoothed estimates are then computed in reverse order, i.e., $\hat{x}(k-1/k)$, $\hat{x}(k-2/k)$, ..., $\hat{x}(j/k)$, ..., $\hat{x}(0/k)$. A block diagram of this smoothing algorithm is shown in Figure 6.1. Thus, the TPBVP of Equations 6.12 through 6.16 can be solved by the Kalman-filter algorithm and the fixed interval smoother if $\hat{R}(k)$ and $\hat{Q}(k)$ are assumed to given.

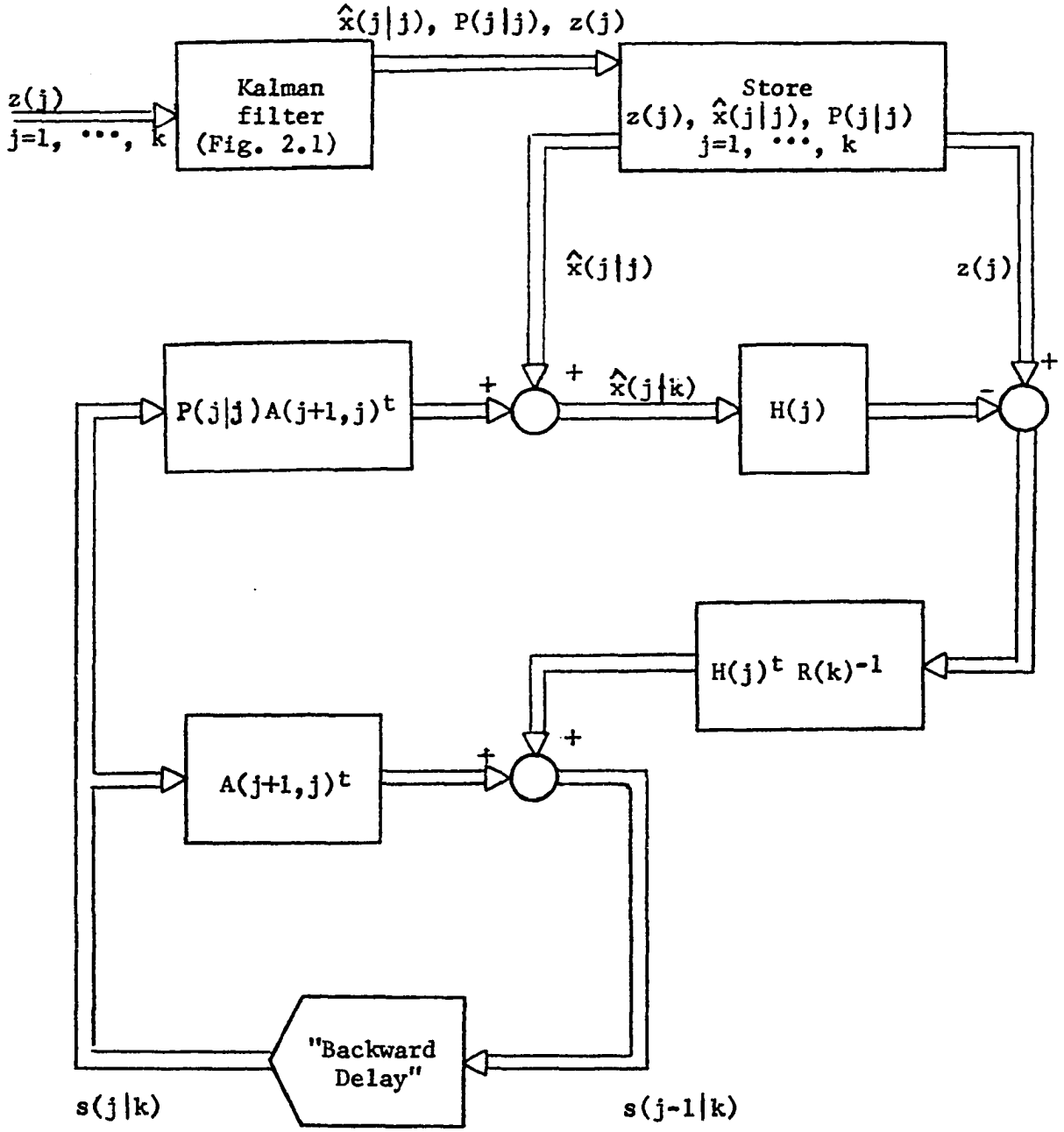

Figure 6.1. A fixed-interval smoothing algorithm

On the other hand, note that if the smoothed estimates, $\hat{x}(0/k)$, $\hat{x}(1/k)$, ..., $\hat{x}(k/k)$, are given, then $\hat{Q}(k)$ and $\hat{R}(k)$ can easily be computed by Equations 6.17 or 6.24 and Equations 6.18 or 6.25. Thus, the overall nonlinear TPBVP is comprised of two sets of equations, each one of which can be easily solved if the correct solution to the other is known. The complication arises because the correct solutions must satisfy both sets of equations simultaneously.

Sage and Husa (5) consider the case where R and Q are characterized by the multivariate uniform distributions. Their resulting necessary conditions are the same as Equations 6.12 through 6.18. Their so-called "correct" solution of these necessary conditions is recursive and consists of the Kalman-filter algorithm along with recursive estimators for R and GQG^t. A block diagram of their overall adaptive estimator is identical in form to Figure 3.1, except that the estimators of R and GQG^t (Equations 3.17 and 3.18) must now be replaced by

$$\hat{R}(k) = \frac{1}{k}\{(k-1)\hat{R}(k-1) + F_R(k)\} \tag{6.30}$$

$$G\hat{Q}(k)G^t = \frac{1}{k}\{(k-1)G\hat{Q}(k-1)G^t + F_Q(k)\} \tag{6.31}$$

where $F_R(k)$ and $F_Q(k)$ are also recursively computed and are functions of $\tilde{z}(k/k-1)$, $\hat{x}(k/k-1)$ and, $\hat{x}(k/k)$. The operation of this estimator is similar to the SH estimator, described

in Chapters III and IV. As each measurement z(k) is processed by the Kalman-filter algorithm, new estimates of R and/or GQG^t are calculated by Equations 6.30 and 6.31. These new estimates of R and GQG^t are then used in the Kalman gain computation for the processing of the next measurement. Heuristic comments, similar to those made about the SH estimator in Chapter IV, can also be made here. However, it will suffice to show that the "solution" of Sage and Husa is not the simultaneous solution of Equations 6.12 through 6.18.

Now when considering the "linear" TPBVP of Equations 6.12 through 6.16, for a given $\hat{R}(k)$ and $\hat{Q}(k)$, the solution is given by the Kalman-filter algorithm and the fixed-interval smoother. The availability of a new measurement at k + 1 requires a new TPBVP to be solved. Now the algorithm of Sage and Husa implies that $\hat{x}(k+1/k+1)$ (the new terminal condition on x with $\hat{R}(k+1)$ and $\hat{Q}(k+1)$ given) is linearly related to $\hat{x}(k/k)$ (the old terminal condition on x with $\hat{R}(k)$ and $\hat{Q}(k)$ given) via the Kalman-filter algorithm. This is <u>not correct</u> because, in the new TPBVP, $\hat{R}(k+1) \neq \hat{R}(k)$ and $\hat{Q}(k+1) \neq \hat{Q}(k)$. This violates the requirement that $\hat{R}(k+1)$ and $\hat{Q}(k+1)$ remain fixed with respect to considering successive TPBVP. The actual relationship is nonlinear and messy. Thus, the simple resetting of the Kalman-filter algorithm with each new estimate of R and Q will not yield the correct

terminal condition on x for each new TPBVP. One way to solve the new TPBVP, with given $\hat{R}(k+1)$ and $\hat{Q}(k+1)$, is to start over from the beginning and use the Kalman-filter (set with $\hat{Q}(k+1)$ and $\hat{R}(k+1)$) to generate a new sequence, $\{\hat{x}(j/j)\}$, $j = 0,1,\ldots,k+1$ and then use the smoother to compute "backwards" the sequence, $\{\hat{x}(j/k+1)\}$, $j = 0,1,\ldots,k$.

A similar objection can be raised with respect to the recursive nature of Equations 6.30 and 6.31. These equations were developed by relating each smoothed estimate, $\hat{x}(j/k+1)$, to its corresponding value in the previous TPBVP, $\hat{x}(j/k)$. This relationship, called the fixed-point smoothing algorithm (24), can be obtained from the solution of Equations 6.12 through 6.16, if the $\hat{R}(k+1)$ and $\hat{Q}(k+1)$ in these equations are equal to the respective $\hat{R}(k)$ and $\hat{Q}(k)$ of the previous TPBVP. Obviously, this is not the case since $\hat{R}(k)$ and $\hat{Q}(k)$ change with each new TPBVP. Therefore, Equations 6.30 and 6.31 are not correct.

Consequently, it can be seen that the overall recursive solution of Sage and Husa (5) is not the simultaneous solution to the overall nonlinear TPBVP because the underlying $\hat{R}(k)$ and $\hat{Q}(k)$ change with each new TPBVP. This destroys the validity of the simple recursive relationship between successive TPBVP.

The MAP algorithm, the method advocated for solving the overall TPBVP at stage k, will now be given. Since this

is an iterative procedure, the notation in Chapter IV will again be used. Therefore, let y_k^j be the value of the quantity, y, generated during the j^{th} iteration in solving the TPBVP with k measurements. To solve the nonlinear TPBVP at stage k, the following procedure could be used. Using guessed values of Q and R, $\hat{Q}(k)_k^o$ and $\hat{R}(k)_k^o$, Equations 6.12 through 6.16 can be solved via the Kalman-filter algorithm and fixed-interval smoothing algorithm to obtain $\hat{x}(j/k)_k^1$, $\hat{w}(j/k)_k^1$, and $s(j/k)_k^1$, $j = 0,1,\ldots,k$. Thus, for a given $\hat{R}(k)_k^o$ and $\hat{Q}(k)_k^o$, the above solution is such that the gradient of the cost function I** (in Equation 6.10) with respect to $\{X(k), W(k-1), S(k-1)\}$, is the null vector. The gradient of I** with respect to Q and R, evaluated at $T(1) = \{\hat{x}(j/k)_k^1, \hat{w}(j/k)_k^1, s(j/k)_k^1, j = 0,1,\ldots,k\}$, will not be the null matrix at $\hat{R}(k)_k^o$, $\hat{Q}(k)_k^o$ because $\hat{R}(k)_k^o$ and $\hat{Q}(k)_k^o$ are not (in general) part of the simultaneous solution to the overall TPBVP. However, for any given T(1), Equations 6.17 (or 6.24) and 6.18 (or 6.25) will give the values of Q and R that set this gradient to the null vector, yielding $\hat{R}(k)_k^1$ and $\hat{Q}(k)_k^1$. Thus, for given values of R and Q, Equations 6.12 through 6.16 are solved for the extremizing values of x(j), w(j), and s(j). Using these values, Equations 6.17 and 6.18 (or Equations 6.24 and 6.25) are solved for new extremizing values of R and Q. This procedure could then be repeated over and over until, hopefully, the successive values of

$$T^*(i) = \{\hat{x}(j/k)^i_k, \hat{w}(j/k)^i_k, s(j/k)^i_k, \hat{R}(k)^i_k, \hat{Q}(k)^i_k,\ j = 0,1,\ldots,k\}$$

will converge to the simultaneous extremum solution of the overall TPBVP at stage k. It is shown in Appendix G that, for a given positive definite R and Q, the extremizing solution of Equations 6.12 through 6.16 actually yields the minimum of I** with respect to X(k). Furthermore, for a given X(k), I** is minimized with respect to Q and R by the estimates in Equations 6.17 (or 6.24) and 6.18 (or 6.25) if the R and Q estimates are p.d., In this case, if I** is convex in X(k), Q, and R, the sequence will converge to the minimum of I**, since with each step of the iteration cycle, one is always determining values which further minimize I**. In Chapter VII, it will be experimentally shown that I** is not always convex.

Even if the algorithm converges to a minimum of I**, it may be that more than one joint minimum exists or that it is located very far from the true values of R and Q. Appendix G shows a method by which one can assess whether, on the average, the a posteriori density (or the equivalent cost function) is a good choice for an estimation criterion.

As another measurement becomes available, a new TPBVP must be solved and the above procedure is repeated. The initial guess for R and Q for each new TPBVP will be the final estimates from the previous TPBVP. Hopefully, after a few measurements have been taken (a few TPBVP have been solved),

it would seem likely that the change, in the final estimates of Q and R from one TPBVP to the next, would be small. Thus, only a small number of iterations would be needed for each TPBVP. In some cases, perhaps only one cycle would be needed.

As described so far, the MAP algorithm is essentially an off-line algorithm because of the increasing memory and increasing computation time needed as the number of measurements increases. However, an approximate procedure, similar to the one advocated for the RF algorithm in Chapter IV, will be able to alleviate this problem. Two types of simplification are proposed. First of all, when the number of measurements becomes large, then the additional information of a few additional measurements for updating the R and Q estimates might not justify the computer cost of computation. Thus, it might be desirable to update the R and Q estimators only every M stages. The recursive Kalman-filter algorithm, set at the last estimates for R and Q, would then be used to estimate the states between the R and Q estimating stages. Thus, the overall TPBVP would be solved only every M stages.

Another simplification might be to use the fixed interval smoother for only N_B steps back, thus fixing the necessary memory and computation time. It may be that fairly good estimates for R and Q may be obtainable with only a few measurements. Also, repeated smoothing of the state at stage j based on more and more measurements will eventually result

in very little improvement in the state estimate. The corresponding terms in the R and Q estimators, that are based on this smoothed estimate, will eventually change very little. Thus, there would be no need to re-smooth back into the distant past. If N measurements were available, then the smoothing would go back to include the state at stage $N - N_B + 1$. The resulting $N - N_B$ smoothed measurement residuals, $\tilde{z}(j/N)$, $j = N-N_B+1,\ldots,N$, would replace their corresponding previously computed values in the estimators for Q and R (Equations 6.17 or 6.24 and Equations 6.18 or 6.25). The new estimates for R and Q would be used in the Kalman-filter algorithm from stage $N - N_B + 1$ to stage $N + M$, whereupon the smoothing would be performed again. This simplification could then be used for on-line applications.

D. Description of the MAP Algorithm

The operation of the MAP algorithm is depicted in Figure 6.2. N is the stage at which the smoothing is to be done, N_B is the number of stages back into the past that the smoothing is carried, J is the total number of iterations at stage N, and M is the number of stages between reprocessing. The general operation of the algorithm will be reviewed, first. Then, some particular modes of operation will be pointed out.

To approximately solve the overall TPBVP at stage N,

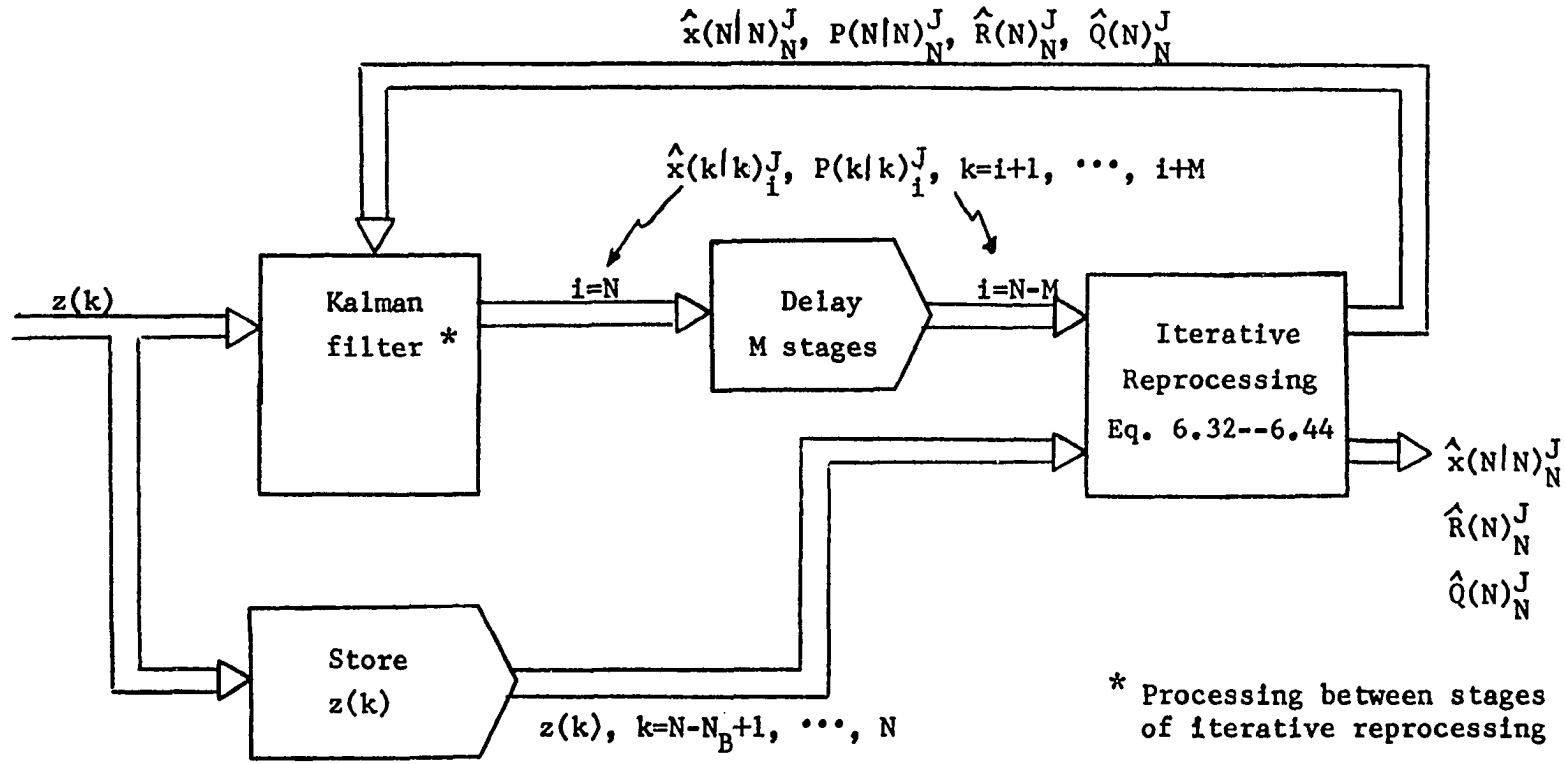

Figure 6.2. Overall MAP algorithm

the Kalman-filter, using the initial estimates $\hat{R}(N)_N^o$ and $\hat{Q}(N)_N^o$, is used to process the measurements from stage $N-M+1$ up to and including stage N. At the same time, the measurements and the other required terms from the Kalman-filter are stored. The fixed-point smoothing algorithm then computes the smoothed state estimates in reverse order back to stage $N-N_B$, enabling new estimates of R and Q to be computed. At that point the Kalman-filter algorithm uses these new estimates, $\hat{R}(N)_N^1$ and $\hat{Q}(N)_N^1$, to reprocess the measurements from stage $N-N_B+1$ to N. If $J>1$, then the smoothing and Kalman-filtering are repeated.

The total set of equations for the j^{th} iteration at stage N will now be given. The j^{th} iteration is also shown in Figure 6.3. The smoothing equations are

$$\hat{x}(k/N)_N^j = \hat{x}(k/k)_N^{j-1} + P(k/k)_N^{j-1} A(k+1,k)^t s(k/N)_N^j \qquad (6.32)$$

$$s(k-1/N)_N^j = A(k+1,k)^t s(k/N)_N^j + H(k)^t [\hat{R}(N)_N^{j-1}]^{-1} [z(k) - H(k)\hat{x}(k/N)_N^j] \qquad (6.33)$$

for $k = N,\ldots,N-N_B+1$ with the boundary condition

$$s(N/N)_N^j = 0 \quad . \qquad (6.34)$$

The estimates of R and Q at the j^{th} iteration are,

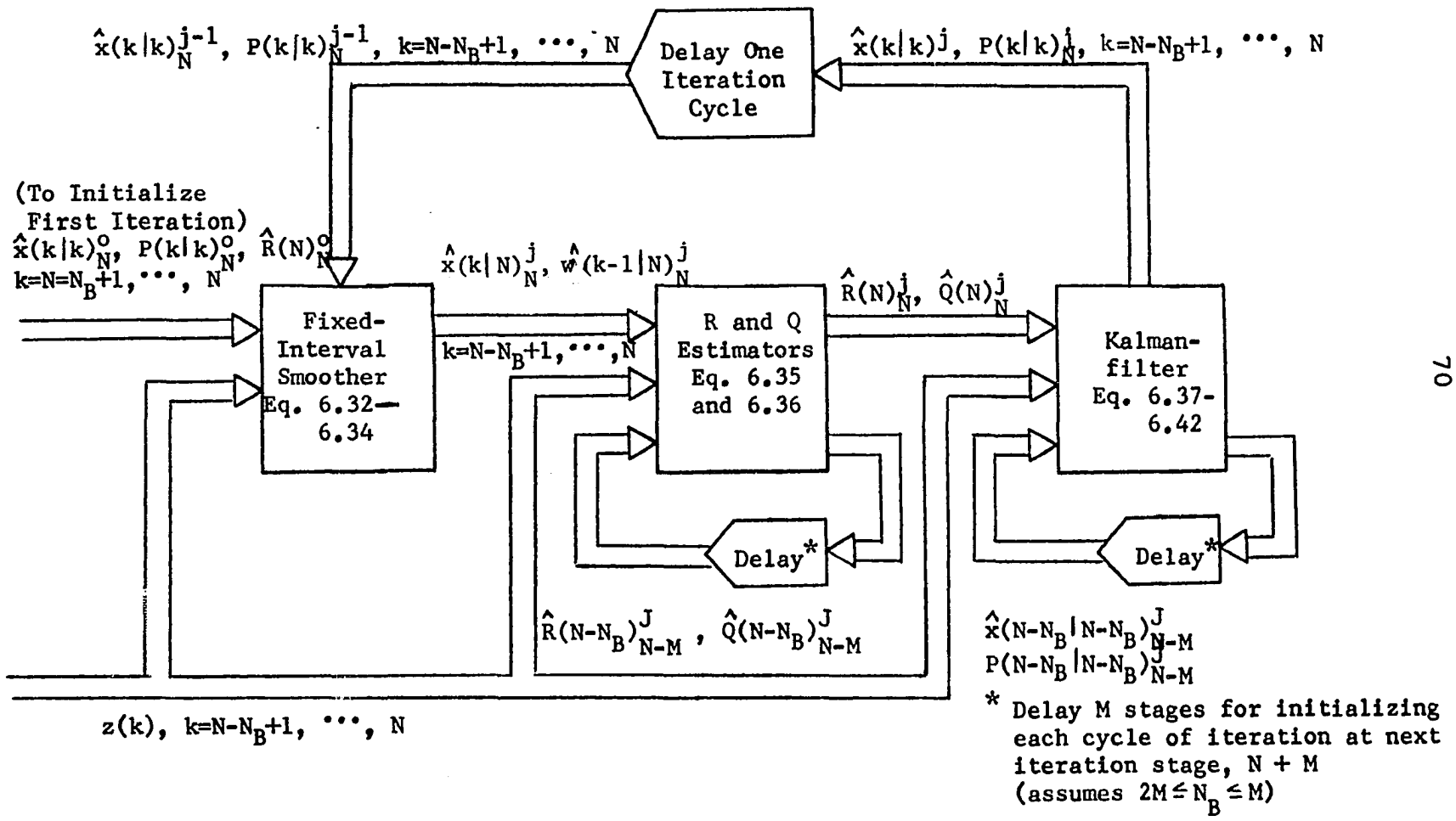

Figure 6.3. Iterative reprocessing flow for MAP algorithm

$$\hat{R}(N)_N^j = \frac{1}{N}\Big\{ \sum_{k=N-N_B+1}^{N} [z(k) - H(k)\hat{x}(k/N)_N^j][z(k) - H(k)\hat{x}(k/N)_N^j]^t$$

$$+ (N-N_B)R(N-N_B)_{N-M}^J\Big\} \tag{6.35}$$

and

$$\hat{Q}(N)_N^j = \frac{1}{N}\Big\{ \sum_{k=N-N_B+1}^{N} \hat{w}(k-1/N)_N^j \hat{w}(k-1/N)_N^{jt} + (N-N_B)\hat{Q}(N-N_B)_{N-M}^J\Big\} \tag{6.36}$$

If the multivariate uniform characterization is assumed to characterize R and/or Q, then the new estimates must be checked for satisfaction of the boundary requirements. The equations for the Kalman-filter algorithm are shown here for completeness to be

$$\hat{x}(k/k-1)_N^j = A(k,k-1)\hat{x}(k-1/k-1)_N^j \tag{6.37}$$

$$\tilde{z}(k/k-1)_N^j = z(k) - H(k)\hat{x}(k/k-1)_N^j \tag{6.38}$$

$$\hat{x}(k/k)_N^j = \hat{x}(k/k-1)_N^j + K(k)_N^j \tilde{z}(k/k-1)_N^j \quad k = N-N_B+1,\ldots,N \tag{6.39}$$

where the Kalman gain computation is

$$P(k/k)_N^j = A(k,k-1)P(k-1/k-1)_N^j A(k,k-1)^t$$

$$+ G(k-1)\hat{Q}(N)_N^j G(k-1)^t \tag{6.40}$$

$$K(k)_N^j = P(k/k-1)_N^j H(k)^t [H(k)P(k/k-1)_N^j H(k)^t + \hat{R}(N)_N^j]^{-1} \tag{6.41}$$

$$P(k/k)_N^j = [I - K(k)_N^j H(k)]P(k/k-1)_N^j \tag{6.42}$$

and the initial values for the Kalman-filter algorithm are

$$\hat{x}(N-N_B/N-N_B)_N^j = \hat{x}(N-N_B/N-N_B)_{N-M}^J \tag{6.43}$$

$$P(N-N_B/N-N_B)_N^j = P(N-N_B/N-N_B)_{N-M}^J \ . \tag{6.44}$$

(It is assumed here that $N_B > M$.) The total processing at stage N entails the recycling through Equations 6.32 through 6.44 for $j = 1,\ldots,J$. The stored quantities, $\hat{x}(k/k)_N^o$, $P(k/k)_N^o$, $k = N-N_B+1,\ldots,N$, are from the Kalman-filter based on the R and Q estimates at the termination of the processing at stage N-M. Therefore,

$$\hat{x}(k/k)_N^o = \text{most updated estimate of } \hat{x}(k/k) \qquad k = N-N_B+1,\ldots,N-M \tag{6.45}$$

$$P(k/k)_N^o = \text{most updated value of } P(k/k) \qquad k = N-N_B+1,\ldots,N-M \tag{6.46}$$

$$\hat{x}(k/k)_N^o = \hat{x}(k/k)_{N-M}^J \qquad k = N-M+1,\ldots,N \tag{6.47}$$

$$P(k/k)_N^o = P(k/k)_{N-M}^J \qquad k = N-M+1,\ldots,N \tag{6.48}$$

$$\hat{R}(N)_N^o = \hat{R}(N-M)_{N-M}^J \ . \tag{6.49}$$

At the end of the processing at stage N, the terminal values, $\hat{x}(N/N)_N^J$, $P(N/N)_N^J$, $\hat{R}(N)_N^J$, $\hat{Q}(N)_N^J$ are used to initialize the Kalman-filter for processing the next M measurements, after which the reprocessing is performed again.

Some particular modes of operation that might be used are as follows. On-line applications would require that N_B be fixed so that the computer memory and computation time is not increasing. One might desire to solve the overall TPBVP at every stage ($M = 1$) when only a few measurements are available since each new measurement would add a significant amount of information about Q and R. As the number of measurements become large, M could be made larger. Because of computer requirements, one might not want to do more than one iteration at each stage N.

In off-line applications, N could be fixed at the total number of available measurements ($M = 0$) and all the measurements would be reprocessed through a number of iterations ($N_B = N, J > 1$).

VII. EXPERIMENTAL RESULTS

In order to demonstrate the efficacy of the adaptive estimation algorithms developed in this dissertation, Monte Carlo computer simulations were made for two system of interest. The two algorithms are compared against each other and also with the adaptive estimator of Sage and Husa (5). Most of the Monte Carlo simulations used a first order system model to save computer costs. However, a second order model with R unknown was simulated to show the advantage of the RF and MAP algorithms.

All the Monte Carlo simulations consisted of 20 sample sequences so that the experimental averages were based on 20 samples. Actually, it would be desirable to have a much larger sampling because a certain amount of statistical irregularity still exists in a sample of this size. However, computer costs prohibited going to any larger size. The quality of the state estimators in the adaptive algorithms was evaluated by computing the experimental mean square state estimation error for each algorithm. To partially offset the restriction to a small sample size, the experimental mean square state estimation error of the Kalman-filter, set to the true values of R and Q, was also calculated and compared to the corresponding mean square error of the adaptive algorithms. Since the state estimating structures of each

algorithm are similar (a Kalman-filter algorithm set to the true or estimated value of R and Q), it was felt that most of the statistical irregularities of the small sample size would be manifested in a similar manner in the results of the experimental averages. Thus, the adaptive estimator whose experimental mean square error was closest to the experimental Kalman-filter mean square error was judged to be the most "optimal" since the Kalman-filter is the minimum mean square error estimator.

One possible and proper way of evaluating the R and Q estimators would be to compare them on the basis of their experimental MSE. However, it was felt that the small sample size would cause statistical irregularities in the MSE curves which would tend to make the convergence properties of the R and Q estimators less apparent. Consequently, each estimator was compared against the most likely values of R and Q when the sequences $\{v(j),w(j)\}$ are known in order to delete the effects of the small sample size as much as possible. In computing these maximum likelihood (ML) estimates of R and Q, it was assumed that the means of the sequences v(j), w(j) were unknown. This was felt to be a more "safe" estimate of R and Q in order to counteract any possibility of a non-zero mean in the random number generator of the Monte Carlo simulation. Consequently, the ML estimates used were (22),

$$\hat{R}_{ML}(k) = \frac{1}{k} \sum_{j=1}^{k} [v(j) - \bar{v}][v(j) - \bar{v}]^t \tag{7.1}$$

$$\hat{Q}_{ML}(k) = \frac{1}{k} \sum_{j=1}^{k} [w(j) - \bar{w}][w(j) - \bar{w}]^t \tag{7.2}$$

where $\bar{v}$ and $\bar{w}$ are the sample means. These estimates must not be confused with the sample covariances used in multivariate statistics which are $\frac{k}{k-1}$ times the terms in Equations 7.1 and 7.2. The comparison between each adaptive estimate of R and Q and the respective ML estimate was accomplished by computing the experimental mean square of the difference between the ML estimate and the adaptive estimate. This was denoted by $MS(\hat{R}_{ML} - \hat{R})$ and $MS(\hat{Q}_{ML} - \hat{Q})$, respectively, where $\hat{R}$ and $\hat{Q}$ are some type of adaptive estimate. An adaptive estimator of R or Q was judged "better" if its above defined experimental mean square value was smaller.

A. A First Order System - R Unknown

Consider the first order system model,

$$x(k) = .95x(k-1) + w(k-1) \tag{7.3}$$

$$z(k) = x(k) + v(k) \tag{7.4}$$

where all the variables are now scalars that satisfy all the assumptions in Chapter II. First of all, let $R = 1$, be the "unknown" measurement variance. Three separate cases are shown in Figures 7.1 through 7.6. They are:

Case 1: $P(0/0) = 1.0, \quad Q = .01$

Case 2: $P(0/0) = 5.0, \quad Q = .01$

Case 3: $P(0/0) = 1.0, \quad Q = .1$.

In these cases, the RF and MAP algorithms used only one cycle of reprocessing at every stage ($J = 1$, $M = 1$ in Equations 4.1 through 4.22 and in Equations 6.32 through 6.49), with all the measurements being reprocessed at each stage $N(N_B = N)$. The initial estimate of R used in all the adaptive algorithms was $\hat{R}(0) = 10$. The MAP algorithm assumed a uniform distribution for R, defined between .1 and 20.

Figure 7.1 shows that state estimator of the MAP algorithm converges very quickly to the Kalman-filter (KF) set to the true value of R. The RF algorithm converges almost as fast, while the SH estimator converges very slowly. A similar situation exists for the R estimators in Figure 7.2. The MAP algorithm tends to track $\hat{R}_{ML}$ very quickly while the RF algorithm is slower but considerably better than the SH estimator. Note, also, that at about the 7th stage, the SH estimator is a little better than the RF algorithm. However, the RF state estimator at this point is doing considerably better. This is because the RF algorithm has repeatedly recorrected itself so that much of the error in the state estimate has been removed, while much of the effects of the initially poor estimate of R is still present in the SH state estimator. The MAP estimator seems to be superior in this

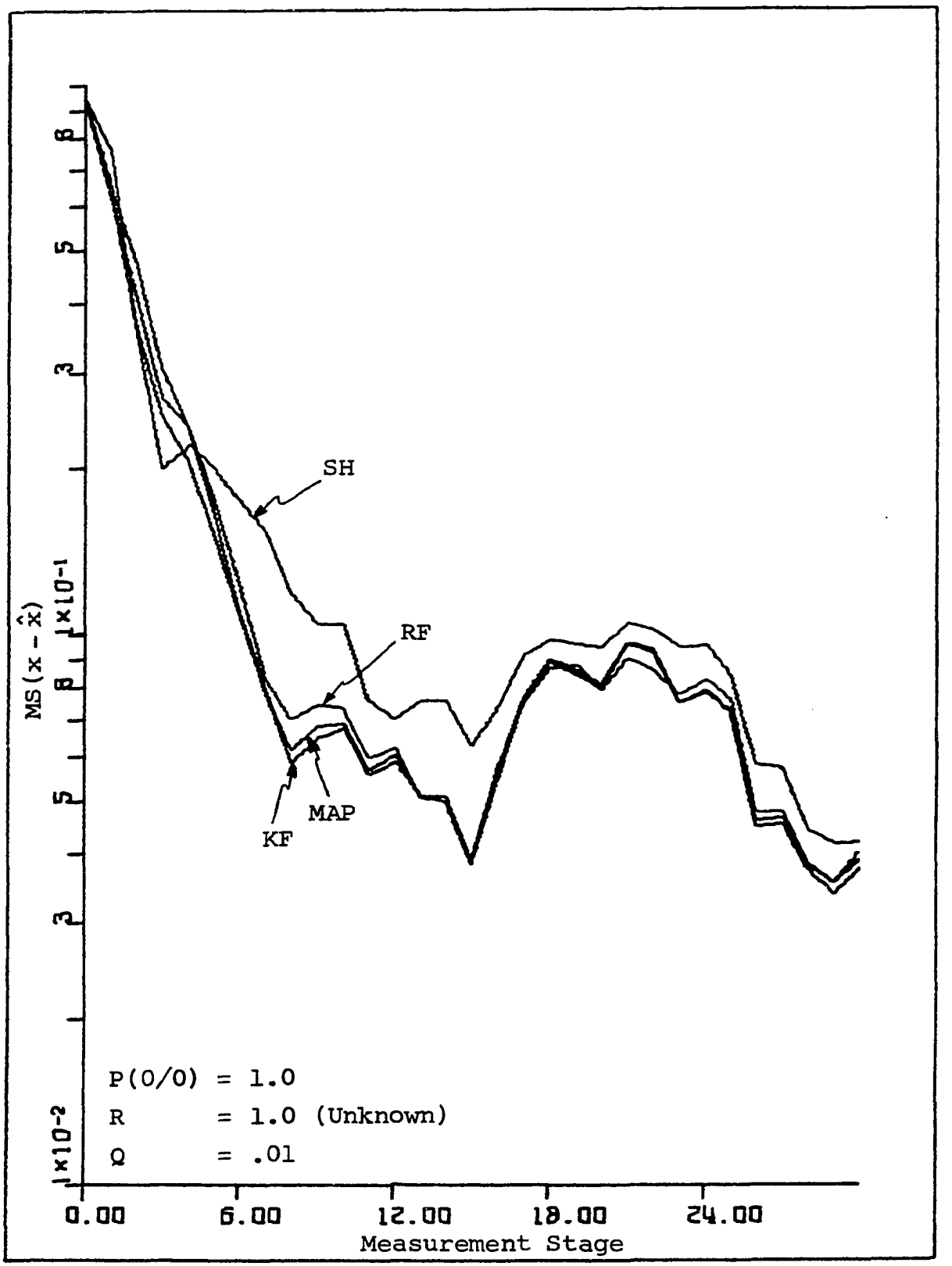

Figure 7.1. Mean square state estimation error for Case 1

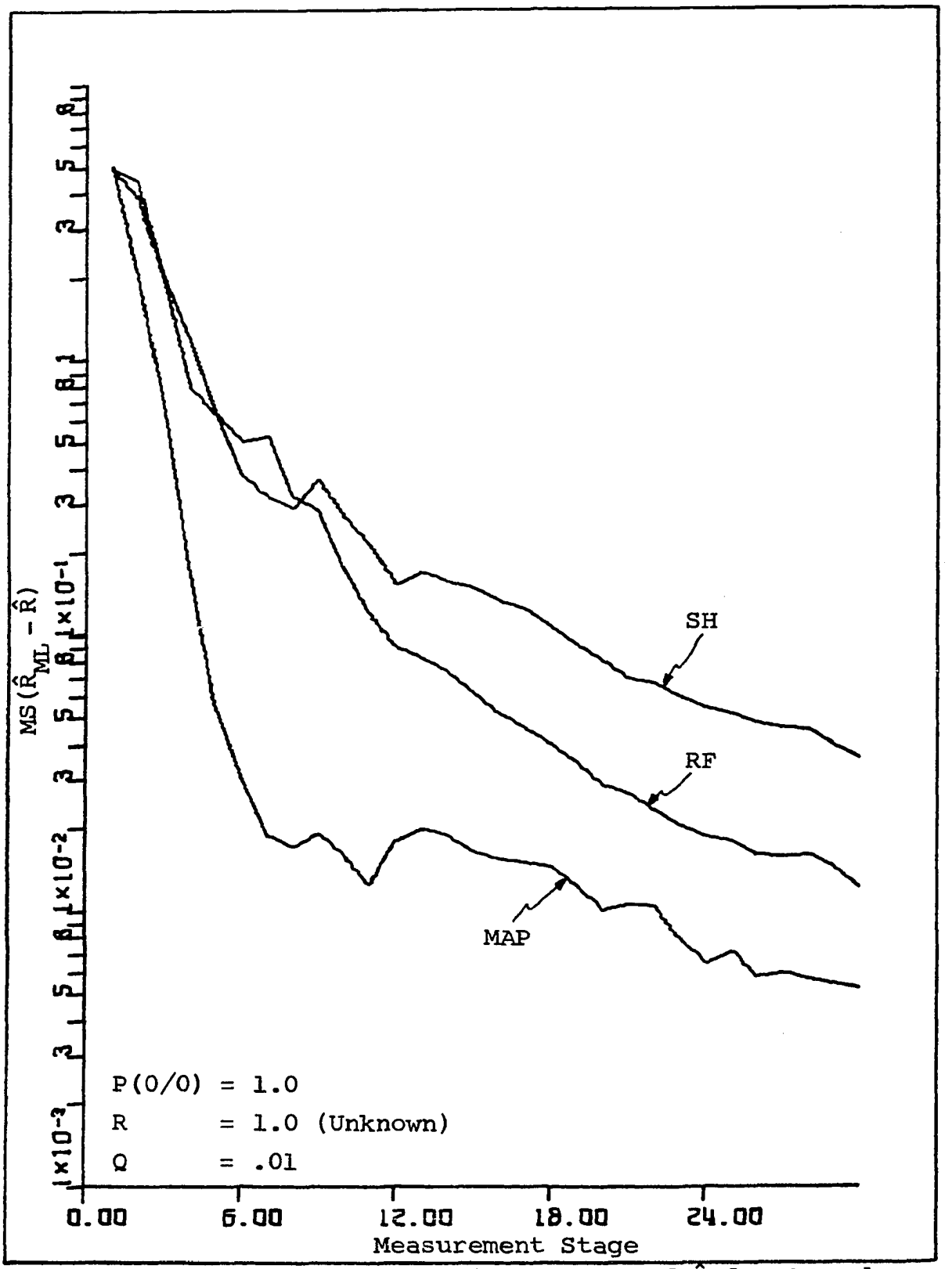

Figure 7.2. Mean square tracking error of $\hat{R}$ for Case 1

case because the R estimator is using more "refined" measurement residuals than in the RF algorithm.

Figures 7.3 and 7.4 show that the results for Case 2 in which the known a priori covariance of the state at the initial stage was increased from 1.0 to 5.0. Here, the MAP estimate remained virtually unchanged while the RF and SH estimators degrade in performance. This is also indicated in Table 7.1, where the performance of the R estimators at the 30^{th} stage are listed. Examination of the R estimators in Equations 3.17, 4.7, and 6.35 show that the SH and RF estimators of R depend directly on P(0/0) while the MAP estimator of R depends on P(0/0) only indirectly through the smoothed state estimates. A large P(0/0) indicates a large uncertainty about the initial state x(0). Consequently, the first few measurement residuals from the Kalman-filter will reflect a

Table 7.1. $MS(\hat{R}_{ML} - \hat{R})$ at the 30^{th} measurement stage for Cases 1, 2 and 3

	Case 1	Case 2	Case 3
SH	.0356	.139	.0604
RF	.0121	.0450	.0387
MAP	.00521	.00536	.0434

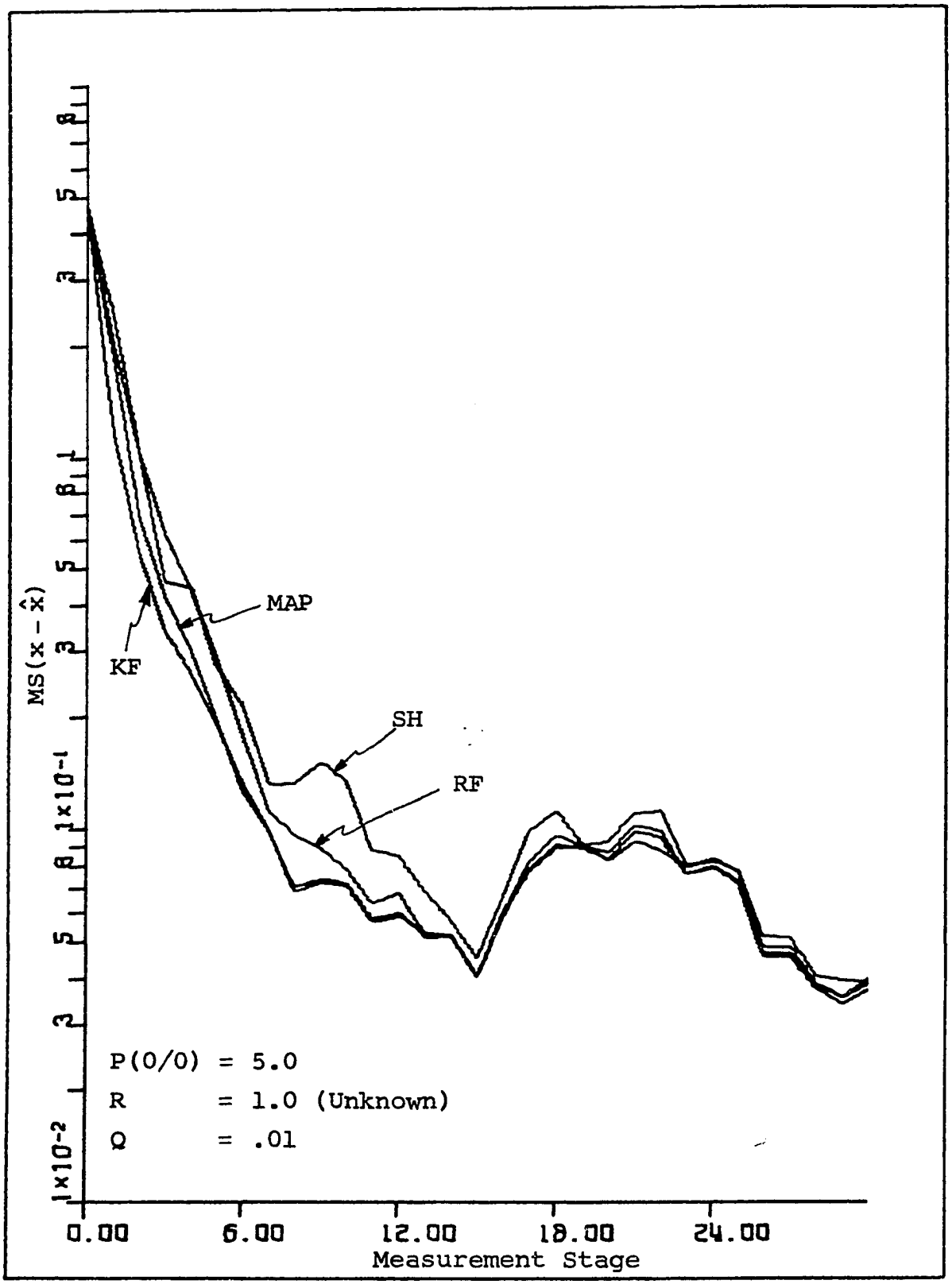

Figure 7.3. Mean square state estimation error for Case 2

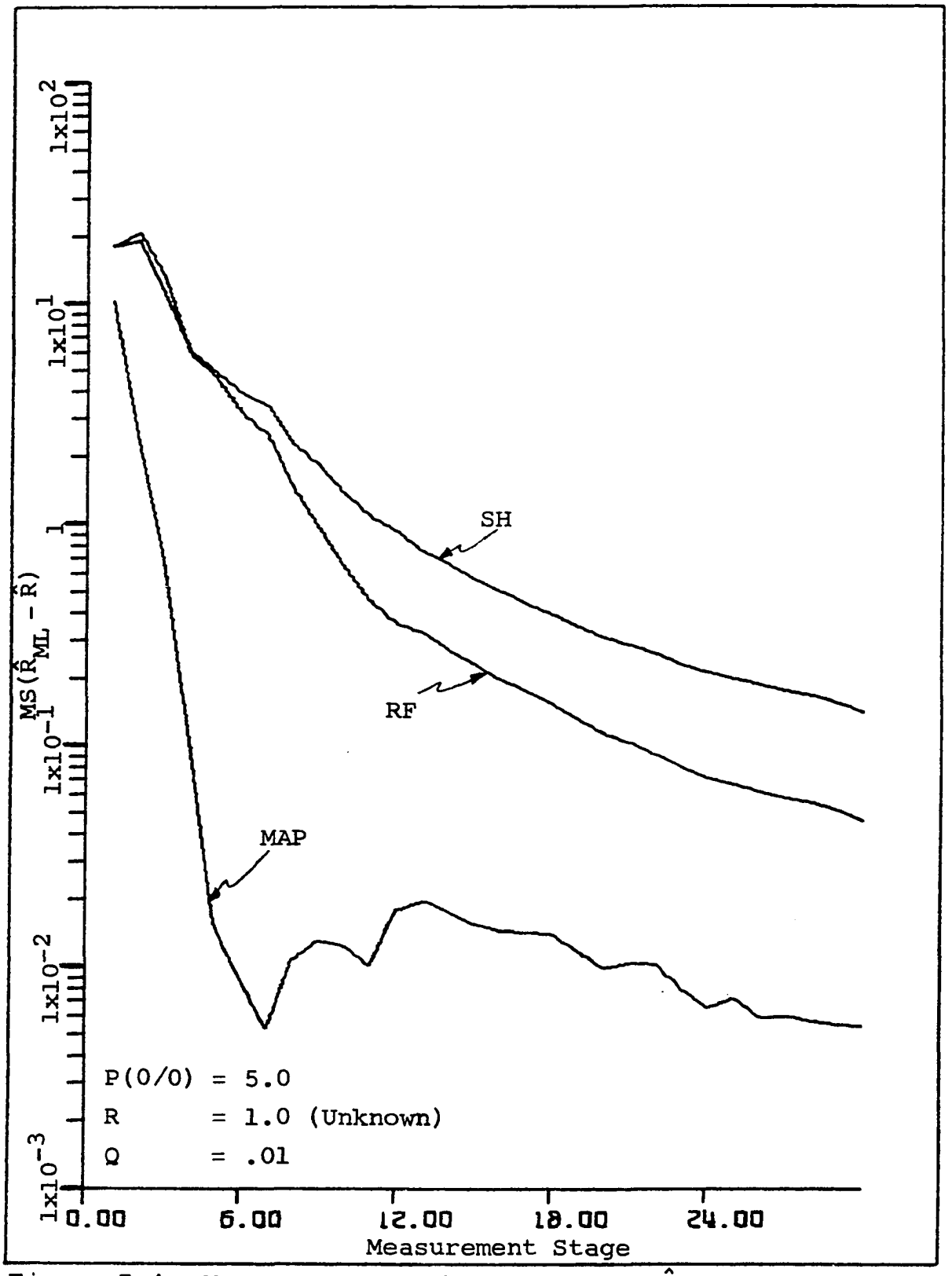

Figure 7.4. Mean square tracking error of $\hat{R}$ for Case 2

large amount of uncertainty due to the initial state. Now, even if the measurement residuals are from the "true Kalman-filter", the first few terms of the RF or SH estimators of R would tend to dominate the rest of the (on the average) smaller residuals for a considerable number of stages. Hence, the R estimators of the SH and RF algorithms will be significantly degraded due to the added uncertainty associated with the first few terms. On the other hand, the smoothed measurement residuals are much less affected by the uncertainty in x(0) because they reflect the extra information obtained from all the available measurements.

Figures 7.5 and 7.6 show the results for Case 3 where the known variance, Q, is ten times larger than in Case 1. The final R estimator performances at stage 30 are also listed for this case in Table 7.1. In Case 3, the MAP estimator has degraded the most while the SH estimator has degraded the least. However, Figure 7.5 still shows that the RF and MAP algorithms tend to track the Kalman-filter MSE better than the SH estimator. This degradation in the RF and MAP algorithms is not too surprising since the Kalman-filter (with the true values for Q and R) does a relatively poorer job than in Case 1. Thus, successive reprocessing does not extract as much information about R because the Kalman-filter algorithm in the RF reprocessor cannot estimate the states as well as in Case 1. Furthermore, the MAP estimator can be

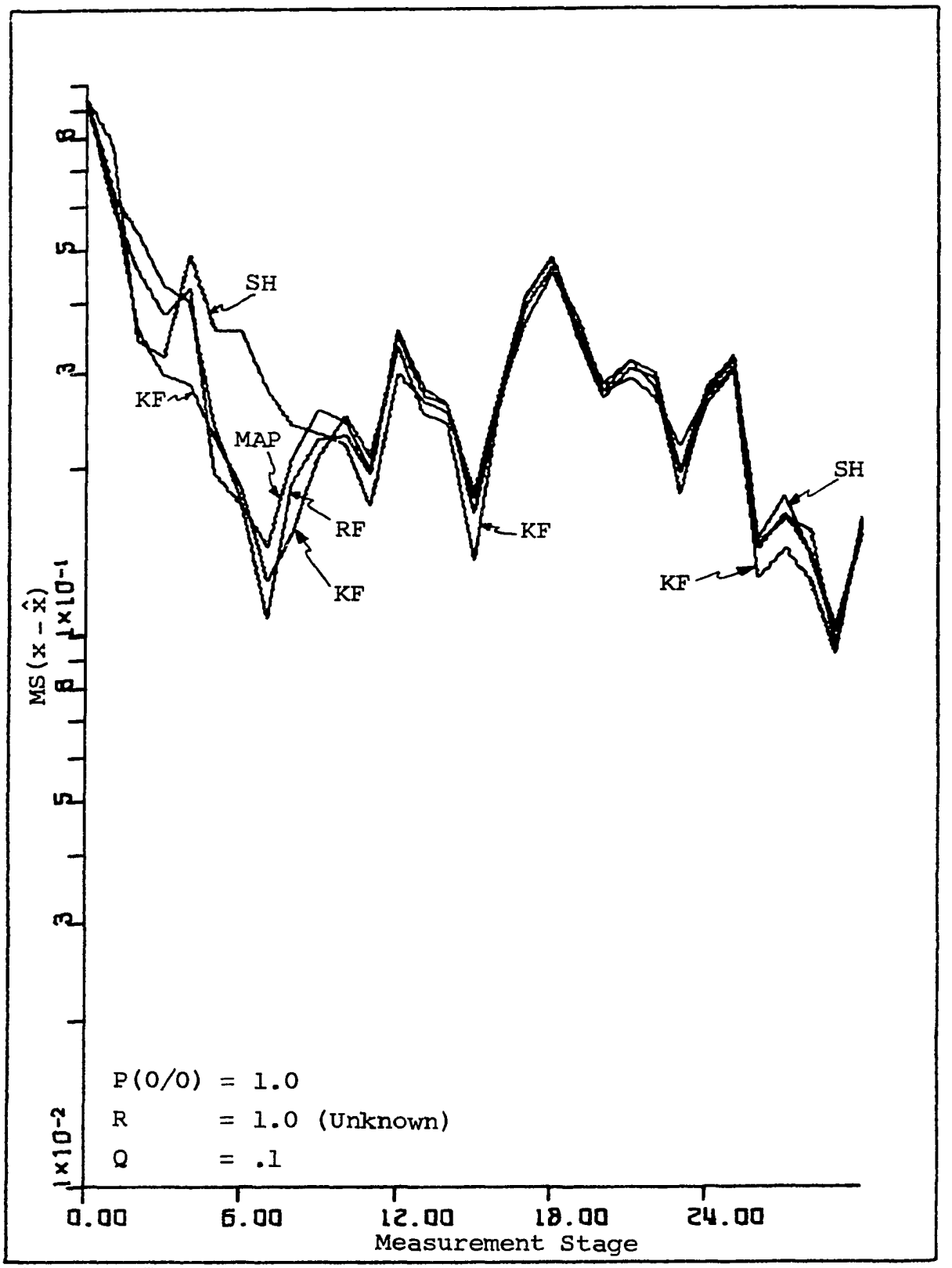

Figure 7.5. Mean square state estimation error for Case 3

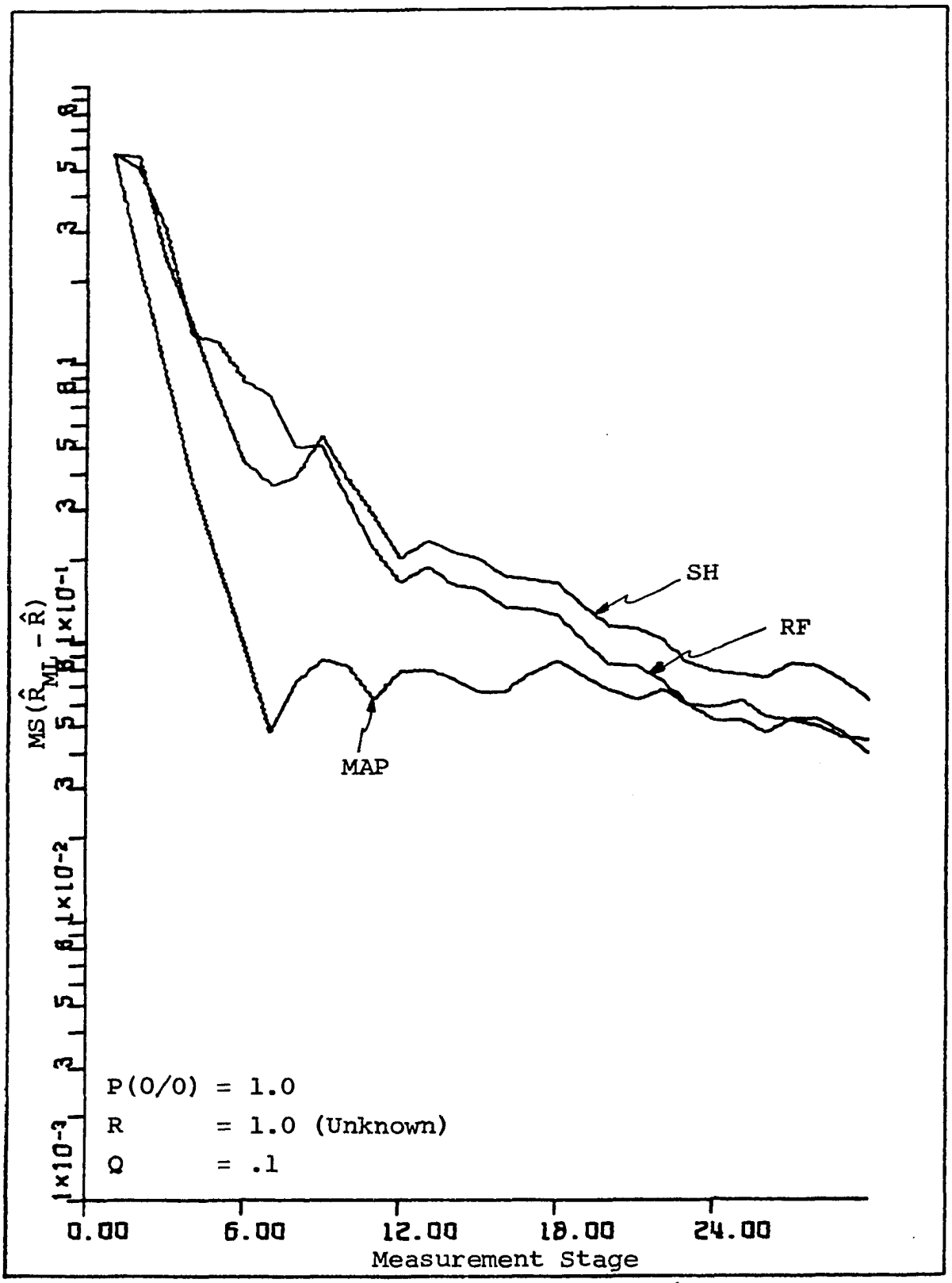

Figure 7.6. Mean square tracking error of $\hat{R}$ for Case 3

expected to degrade even more because both the Kalman-filter and smoothing algorithms will have degraded in their estimating ability, thus, compounding their effect on the resulting MAP estimates. The SH estimator is least affected by the increase in Q because it does the least amount of processing.

Additional cases were investigated where P(0/0) = 1, but Q was made increasingly larger. The MAP estimator became progressively worse until at Q = 1.0, it was not able to estimate Q at all. The RF and SH estimators progressively degraded but were still able to estimate R (although poorly) at Q = 5. In these cases, the RF estimator was still superior to the SH estimator, but the difference between them diminished as Q became larger. Also, the estimates seemed to be converging, although very slowly.

To gain more insight as to why the MAP estimator of R deteriorates as Q increases, it is shown in Appendix H that even if the R estimator is using the optimum smoothed estimates in the measurement residuals, the average value of the R estimator would be

$$E\{\hat{R}(k)\} = R_t - \frac{1}{k} \sum_{j=1}^{k} H(j)P(j/k)H(j) \qquad (7.5)$$

where P(j/k) is the fixed-interval smoothing error covariance and R_t is the true value of R. Now when Q gets larger, there is less correlation between successive states. Thus, there will be less correlation between the measurement z(k) and the

state x(j). The result is that the smoothing error covariance will be larger because the measurement at k will contain less information about the state at j. With P(j/k) being larger, the mean of the R estimator degrades farther from the true value of R. Therefore, unless P(j/k) becomes small, the mean of $\hat{R}(k)$ will never be close to R_t. Apparently, from Equations 7.2, 7.4 and 7.6, the smoothing covariance must decrease very rapidly to enable the superior estimation of R over the RF and SH algorithms.

A possibility for future investigation would be to subtract out the mean values from the R estimate. Perhaps even better convergence properties can be obtained.

So far, the RF and MAP estimators have been used in their most ideal mode, i.e. that of reprocessing all the measurements each time a new measurement was available. Consequently, they can only be considered as off-line algorithms. Figures 7.7 and 7.8 show the performance of suboptimal on-line RF and MAP algorithms where only the N_B most recent measurements were processed at each stage. It can be seen that slightly degraded performance was achieved for the MAP estimator, going only 5 stages back, and the RF estimator, going 10 steps back. It seems that the ability of the MAP estimator to quickly track $\hat{R}_{ML}$ enables only a few (5) of the past measurements to be reprocessed.

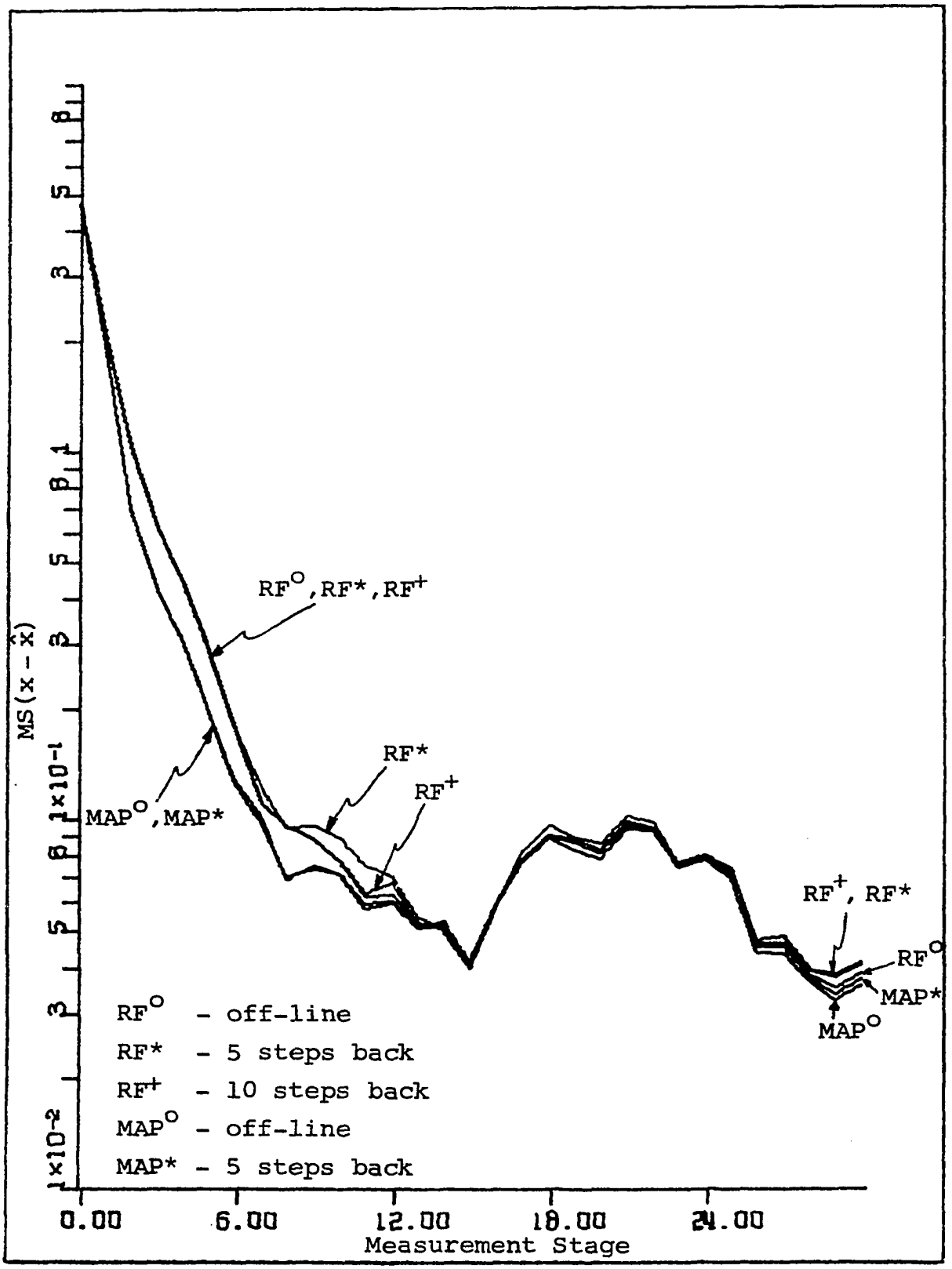

Figure 7.7. Mean square state estimation error for comparison of on-line and off-line RF and MAP algorithms

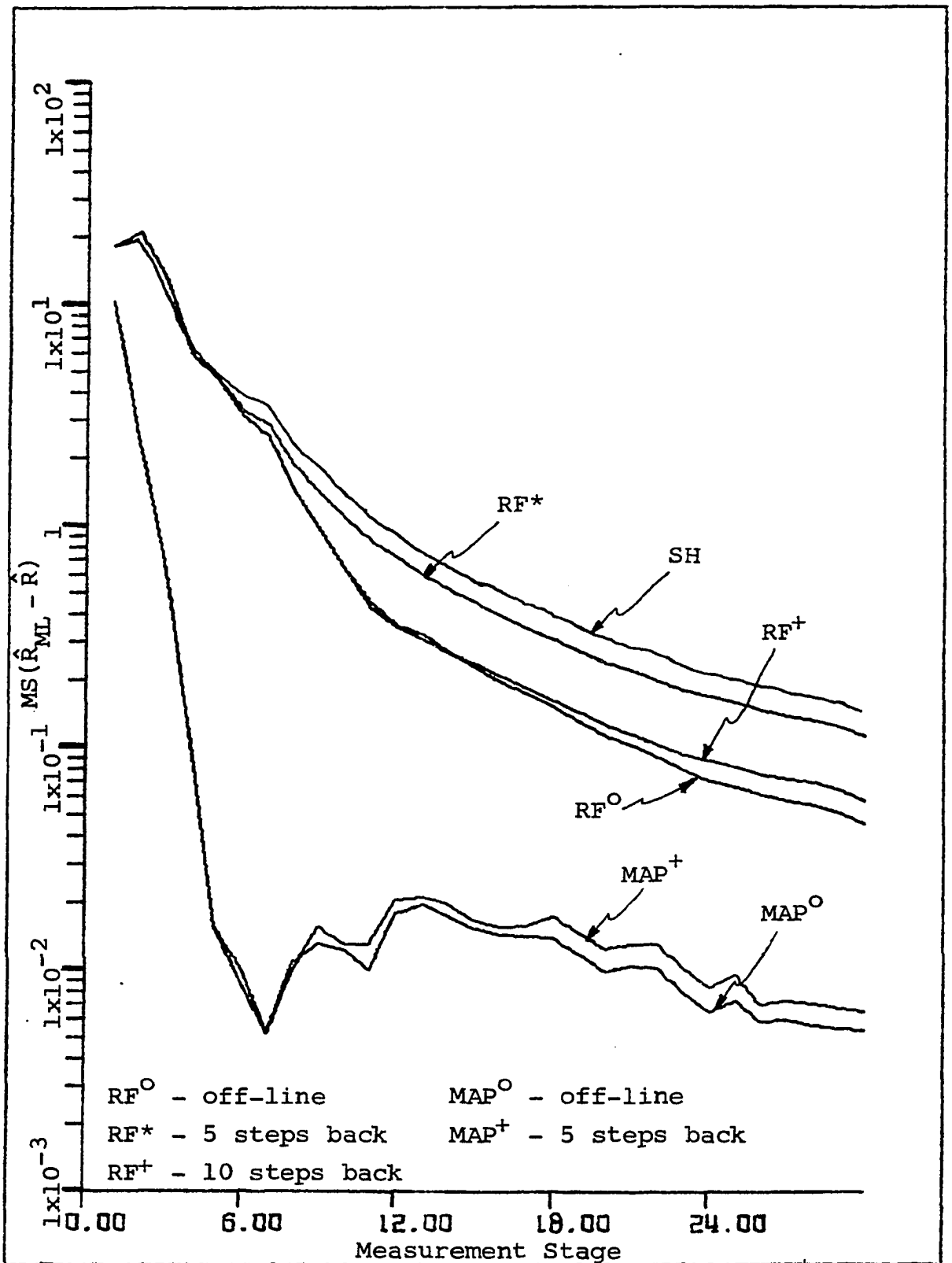

Figure 7.8. Mean square tracking error of $\hat{R}$ for comparison of on-line and off-line RF and MAP algorithms

B. Maximization of the A Posteriori Density

In Chapters V and VI, the MAP algorithm was developed to maximize the a posteriori density $P[X(k),Q,R/Z(k)]$ or equivalently, minimize the cost functions I* in Equation 6.7 or I* in Equation 6.22. A graph of the "partially minimized" cost function of Equation 6.7 is shown in Figure 7.9 for a particular sample function from the system of Case 3. $I_{MAP}(R)$ is the value of I* when it has been minimized with respect to X(k) for each given value of R. Thus, for a given R, $I_{MAP}(R)$ contains the fixed interval smoothed estimated of X(k). It can be seen from Figure 7.9 that the cost functions tend to become more peaked as the number of measurements becomes larger. Table 7.2 shows that convergence of the MAP algorithm to the minimum values of the cost functions in Figure 7.9. In this case the MAP algorithm was iterated for $J = 7$ times at stages 5, 10, 15, and 20 with two different initial guesses for R, 10. and 0.1. The true value for R was 1. It can be seen that the MAP algorithm converges faster when the cost function is more peaked. Also, the maximum of the a posteriori density (minimum of $I_{MAP}(R)$) tends to be closer to $\hat{R}_{ML}$ as k gets larger.

Additional plots of $I_{MAP}(R)$ were made for larger values of Q (the true value of R remained at 1). As Q became larger, $I_{MAP}(R)$ became less peaked until at $Q = 1$, a relative maximum occurred close to $R = 0$. At $Q = 5$, $I_{MAP}(R)$ was concave

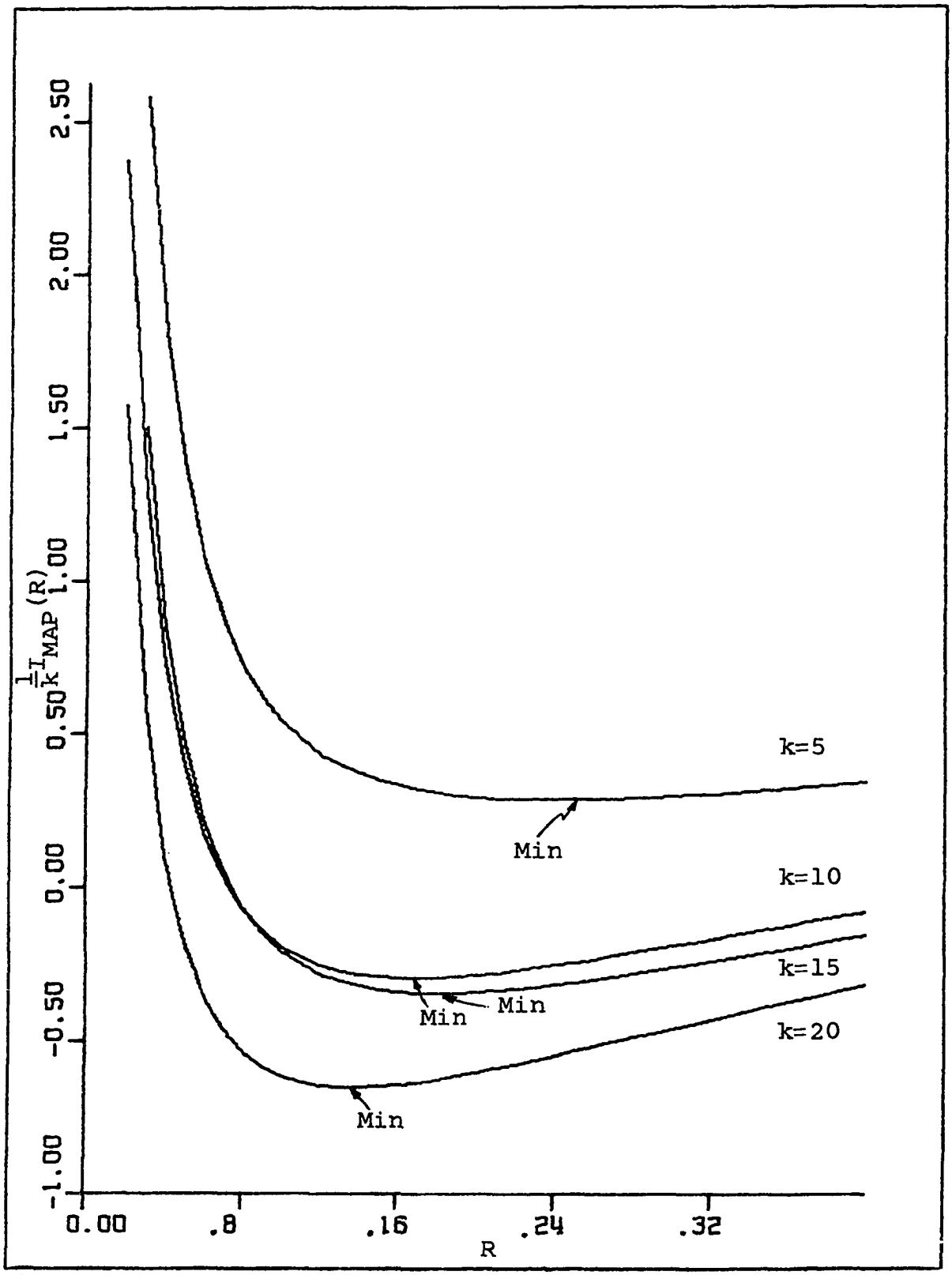

Figure 7.9. MAP cost function vs. number of measurements for Case 3

Table 7.2. Convergence of the MAP algorithm in the successive iteration mode

Iterations j	$\hat{R}(5)^j_5$	$\hat{R}(5)^j_5$	$\hat{R}(10)^j_{10}$	$\hat{R}(10)^j_{10}$	$\hat{R}(15)^j_{15}$	$\hat{R}(15)^j_{15}$	$\hat{R}(20)^j_{20}$	$\hat{R}(20)^j_{20}$
0	10.000	.100	10.000	.100	10.000	.100	10.000	.100
1	5.057	.763	3.240	.566	3.000	.682	2.367	.559
2	3.592	1.555	2.060	1.208	2.094	1.363	1.574	1.071
3	2.971	1.967	1.775	1.463	1.874	1.698	1.405	1.268
4	2.679	2.338	1.695	1.656	1.812	1.782	1.362	1.345
5	2.536	2.366	1.672	1.660	1.793	1.784	1.351	1.346
6	2.465	2.380	1.665	1.661	1.789	1.785	1.348	1.347
7	2.429	2.387	1.663	1.662	1.786	1.786	1.347	1.347

Number of measurements, k	5	10	15	20
R which minimizes $I_{MAP}(R)$	2.4	1.65	1.78	1.35
$\hat{R}_{ML}$	1.278	1.516	1.520	1.282

with the only minimum at R = 0. Thus, the MAP estimator is useless for this case when Q is greater than 0.5.

C. Maximization of the Likelihood Function of R by the RF Algorithm

The likelihood function of R is

$$L(R) = \ln\ P[Z(k)/R] \quad . \tag{7.6}$$

When it is evaluated in terms of the system in Equations 2.1 and 2.2 (neglecting constant terms),

$$-L(R) = \frac{1}{2} \sum_{j=1}^{k} \ln|H(j)P(j/j-1)H(j)^t + R| + \frac{1}{2} \sum_{j=1}^{k} \tilde{z}(j/j-1)^t [H(j)P(j/j-1)H(j)^t + R]^{-1} \tilde{z}(j/j-1) \tag{7.7}$$

where $\tilde{z}(j/j-1)$ and $P(j/j-1)$ are the measurement residuals and covariance matrix from a Kalman-filter set to the given value of R. Now if the system is stationary and k is assumed large enough so that the Kalman-filter is in the steady state, then Kashyap (13) shows that the estimator for R is,

$$\hat{R}(k)_{ss} = \frac{1}{k} \sum_{j=1}^{k} \tilde{z}(j/j-1)\tilde{z}(j/j-1)^t - HP^*_{ss}H^t \quad . \tag{7.8}$$

Note that this is the same estimator that would result if the same steady-state assumptions were applied to the RF estimator of R. Consequently, as k becomes large, the RF algorithm,

when estimating R is conjectured to be maximizing the likelihood of R if the system is stationary. Figure 7.10 shows plots of $-\frac{1}{k}L(R)$ for k = 5, 10, 15, and 20. Table 7.3 shows the convergence of the RF algorithm when it is used in the successive iteration mode (J = 4) at the stages k = 5, 10, 15, and 20. The particular sample sequence used is the same as used in generating the likelihood curves in Figure 7.10 and was based on the system in Case 3. Notice that as k increases, the successive iteration at stage k converges to a value that is closer to the maximum of L(R), verifying the above conjecture development.

Additional plots of $-\frac{1}{k}L(R)$ were made for larger values of Q. Although they became less peaked as Q became larger, the plot at Q = 5, still possessed a global minimum.

D. A First Order System - Q Unknown and R and Q Unknown

Due to computer costs, a thorough Monte Carlo examination of these cases was not accomplished. Most of the reported results will be based on single sample sequence runs. All runs use a uniform characterization for Q and R. First consider where only Q in unknown. With the first order system in Equations 7.3 and 7.4, and with R = .01, P(0/0) = 5, and the unknown true value of Q = 1, the Q estimators of the SH and RF (single iteration, processing all the measurements at each stage) algorithms yielded almost identical performances with

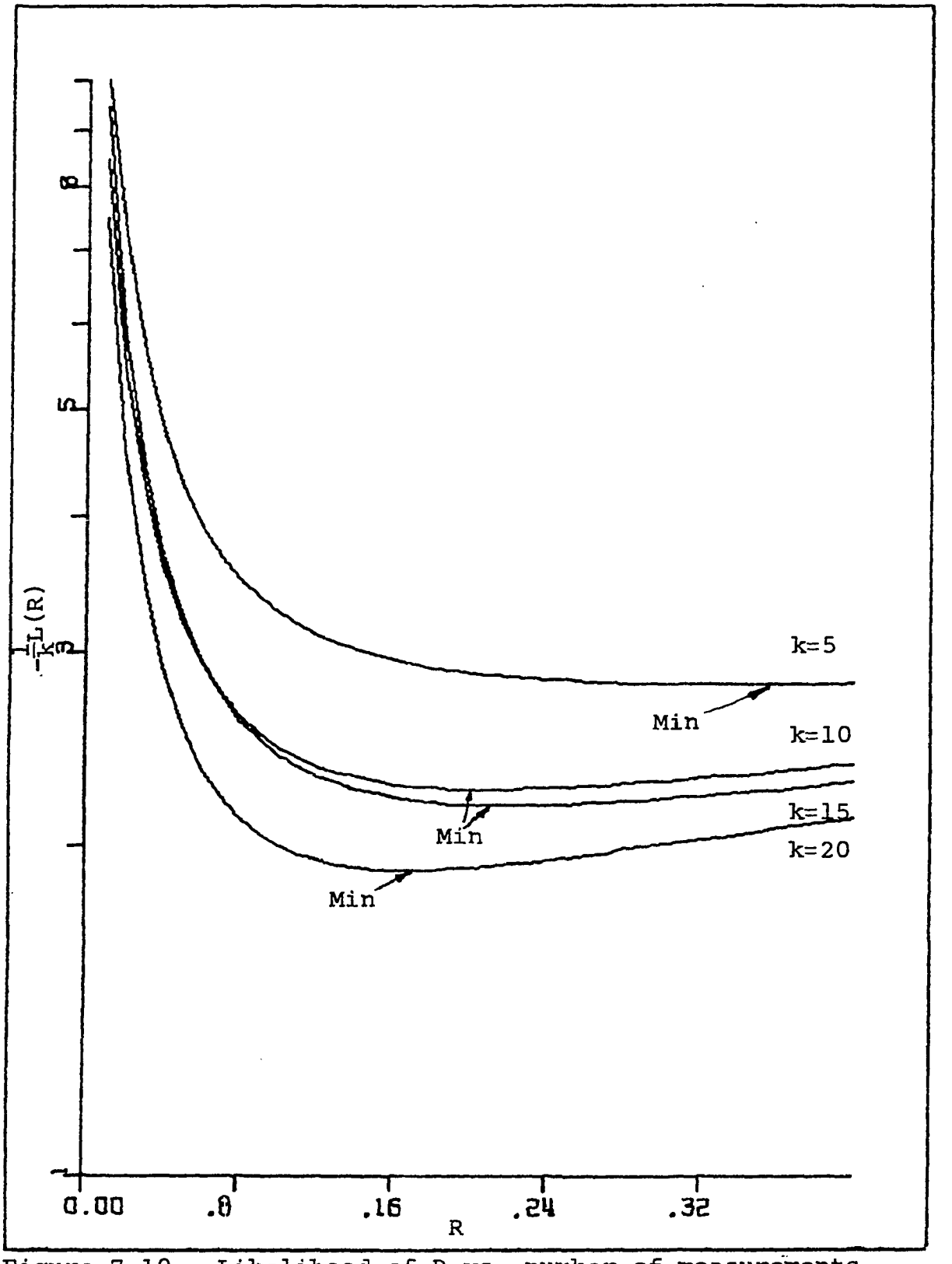

Figure 7.10. Likelihood of R vs. number of measurements

Table 7.3. Convergence of the RF algorithm in the successive iteration mode

Iterations j	$\hat{R}(5)^j_5$	$\hat{R}(5)^j_5$	$\hat{R}(10)^j_{10}$	$\hat{R}(10)^j_{10}$	$\hat{R}(15)^j_{15}$	$\hat{R}(15)^j_{15}$	$\hat{R}(20)^j_{20}$	$\hat{R}(20)^j_{20}$
0	10.0	.10	10.0	.10	10.0	.10	10.0	.10
1	6.383	3.199	4.035	2.398	3.281	2.630	2.424	2.124
2	5.832	4.927	3.299	2.974	2.867	2.796	2.108	2.084
3	5.714	5.492	3.162	3.097	2.823	2.815	2.083	2.081
4	5.687	5.635	3.135	3.122	2.818	2.817	2.081	2.081

Number of measurements, k	5	10	15	20
R which minimizes L(R)	3.3	2.1	2.2	1.7
$\hat{R}_{ML}$	1.278	1.516	1.520	1.282

$\hat{Q}(30)_{SH} = 1.735$, $\hat{Q}(30)_{RF} = 1.731$ and $\hat{Q}_{ML}(30) = .9721$. Here, then, the reprocessing yielded no improvement. Since R was small relative to P(0/0) and Q, the sensitivity of the gain K(k) and the a posteriori covariance P(k/k) to changes in the estimates of Q was very small, so that reprocessing with "better" estimates of Q did not result in any significant change in measurement residuals on the computed covariances. The MAP estimator was able to do a better job ($\hat{Q}(30)_{MAP} = 1.038$) because it used smoothed measurement residuals.

A Monte Carlo run with $Q_t = 5.$, $R = 1.$, and $P(0/0) = 1$, showed that the MAP estimator of Q was the worst while the SH and RF estimators yielded about the same results. The reason for the RF estimator not being any better than the SH estimator is probably due partially to the insensitivity of the Kalman gain and a posterior covariance equations to changes in $\hat{Q}(k)$ when R is small. Also, the measurement is now relatively more "noisy" than in the previous case so that the state estimates will always be relatively poor even if Q is known.

Similarly, as in Section B of this chapter, plots of the "partially optimized" cost function, $I_{MAP}(Q)$, was generated for R = .01, .1, and 1. when $Q_t = 1$. For R = .01 and .1, both plots were convex possessing minimums at Q= 1.0 and 1.2 respectively. However, for R = 1., the plot was concave

with the only minimum at $Q = 0$.

In considering the case where both R and Q were unknown, it was seen that the MAP estimator could not be used in this situation because each covariance estimator was a good estimator only when the other covariance was relatively small. So even at best, one estimator might be quite good while the other would be very poor. A Monte Carlo simulation was run for Case 1 where both Q and R were to be estimated by the SH and RF estimators. Both adaptive algorithms were able to estimate Q and R but there was little difference in their results.

E. Comparison of the Inverted Wishart and Uniform Distribution Characterizations for MAP Estimators of R

Figures 7.11 and 7.12 show the performance of the MAP estimators of R when the uniform and inverted Wishart densities are the a priori characterizations. These plots are from a single sample-sequence of the first order system in Equations 7.3 and 7.4, where $Q = .01$, $P(0/0) = 1.0$, and $R_t = 1.0$. Only one cycle of reprocessing all the measurements was used at each stage ($N_B = k$, $j = 1$, $M = 1$). Now it can be seen from Figure 7.11 that the a priori uniform characterization of R permits a wide range of initial guesses that all quickly converge to the same result. This is certainly desirable when very little information of the true value of R is available.

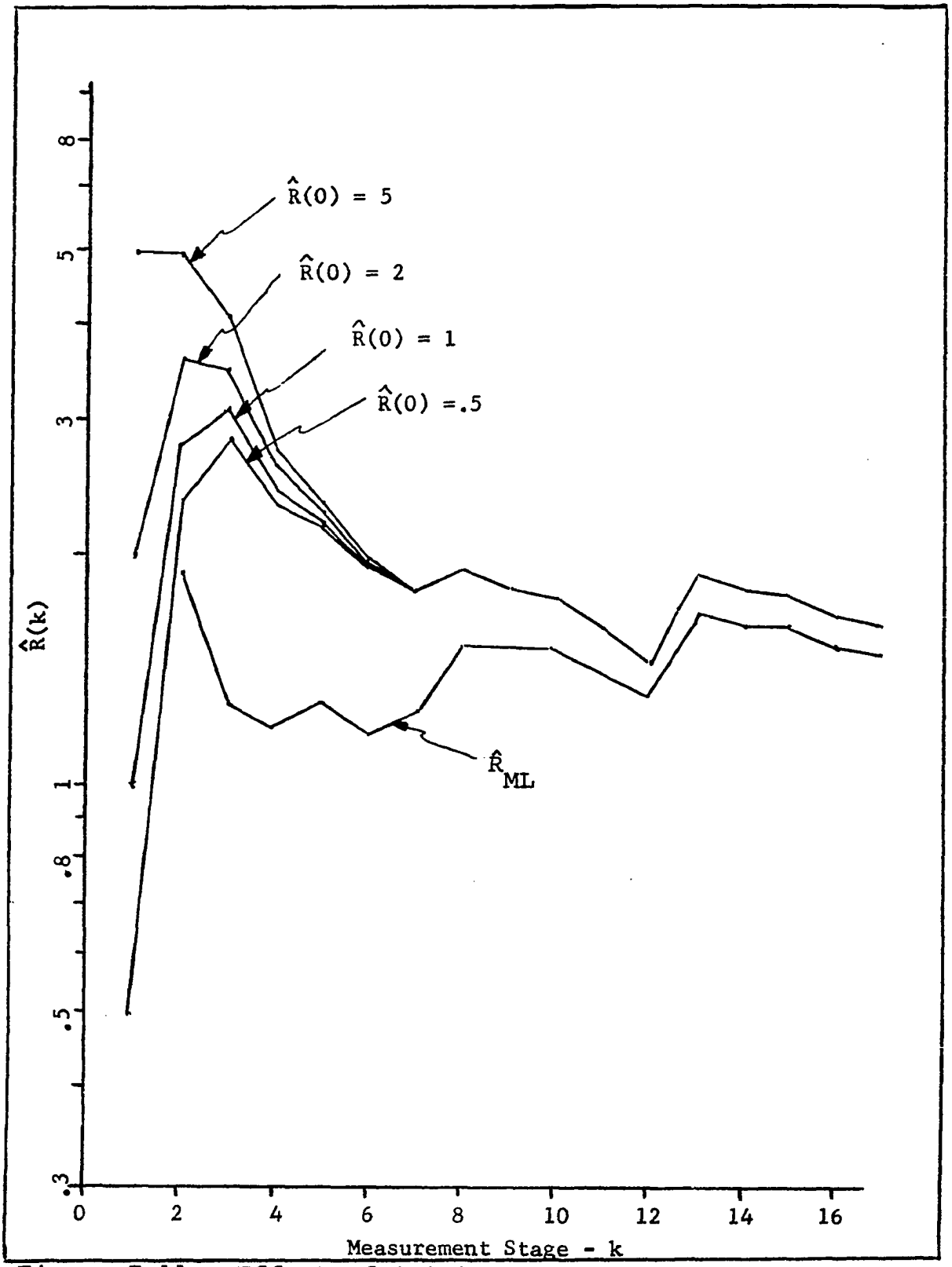

Figure 7.11. Effect of initial guess using uniform characterization of R

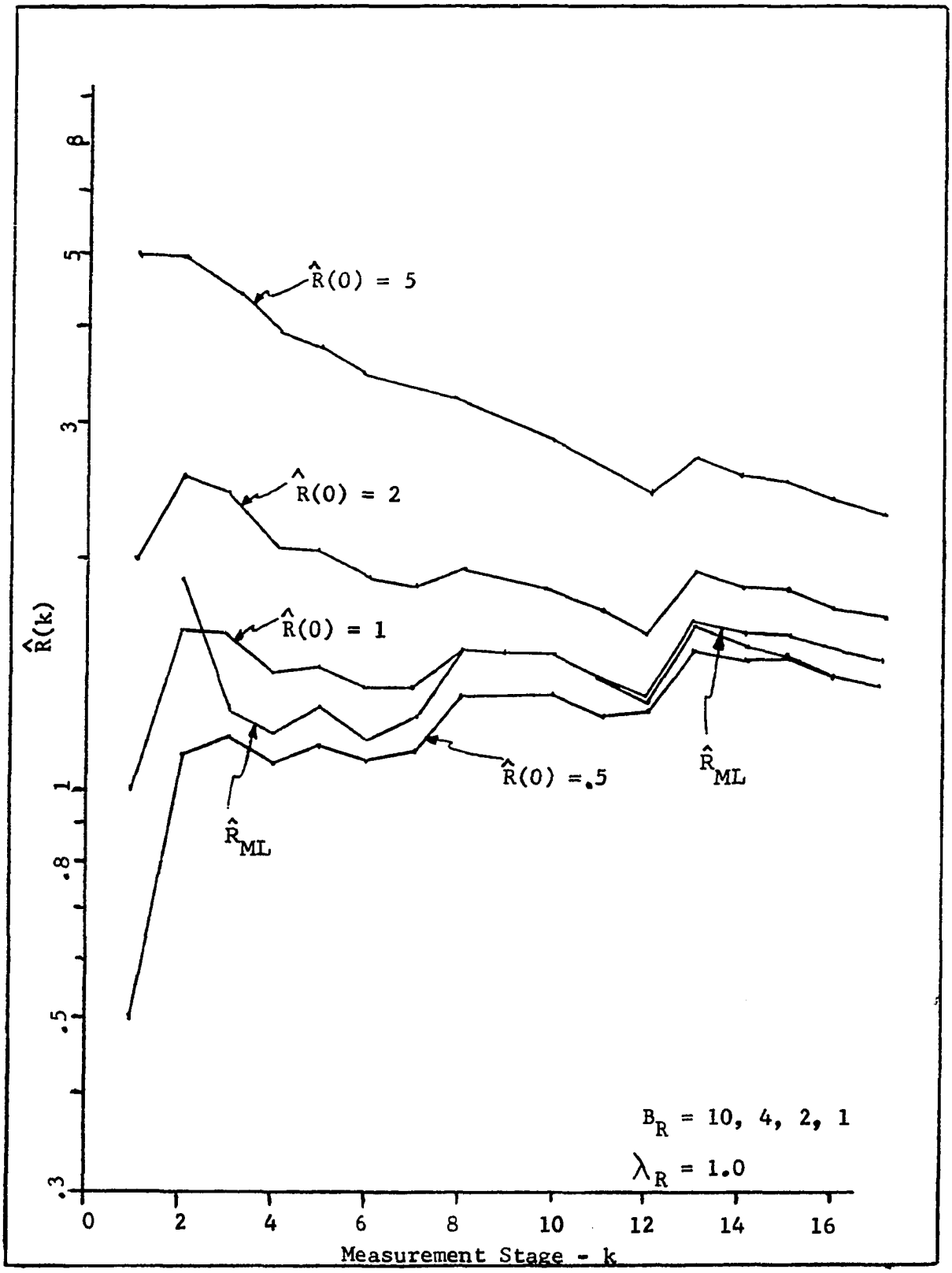

Figure 7.12. Effect of a priori inverted Wishart characterization of R

In Figure 7.12, the parameter λ_R (see Equation 5.17) was fixed at 1. with $B_R = 10, 4, 2, 1$, yielding modes of the a priori densities to be 5, 2, 1, and .5. The modes of the a priori densities were the initial values in the MAP algorithm. It can be seen that if one has a good initial guess of R (i.e., $\hat{R}(0) = .5, 1., 2.$) then initially this yields better results than does the uniform characterization. However, as k gets large, the R estimator of the uniform characterization performs as well, if not better. Also, if a bad guess is made ($\hat{R}(0) = 5.$), then the estimator converges very slowly. Consequently, unless a very good initial guess is known, it seems safer to use the uniform density characterization.

F. A Second Order System with R Unknown

Consider the second order model,

$$x(k) = \begin{bmatrix} .95 & .05 \\ 0 & .995 \end{bmatrix} x(k-1) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(k-1) \qquad (7.9)$$

$$z(k) = \begin{bmatrix} .5 & .1 \\ 0 & 1. \end{bmatrix} x(k) + v(k) \qquad (7.10)$$

where

$$Q = \begin{bmatrix} .1 & .005 \\ .005 & .01 \end{bmatrix} \qquad (7.11)$$

$$R_t = \begin{bmatrix} 1. & .5 \\ .5 & 1. \end{bmatrix} \tag{7.12}$$

with

$$P(0/0) = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} . \tag{7.13}$$

Figures 7.13 through 7.18 show the Monte Carlo results of this case. Again the MAP and RF algorithms reprocessed all the measurements at each stage, only once. The initial guess for R used in all the estimators was

$$\hat{R}(0) = \begin{bmatrix} 2. & 0 \\ 0 & 2. \end{bmatrix} . \tag{7.14}$$

Two different MAP estimators were used. One used a trivariate uniform density for R as

$$P[R] = \begin{cases} \text{Constant} \begin{cases} .02 \leq r_{11} \leq 20. \\ .02 \leq r_{22} \leq 20. \\ r_{12}^2 \leq .9\, r_{11}\, r_{22} \end{cases} \\ 0 \qquad \text{otherwise} \end{cases} . \tag{7.15}$$

The same bounds were used in the SH and RF estimators. The other MAP estimator used an inverted Wishart distribution of Equation 5.17 with

$$B_R = \begin{bmatrix} 4. & 0 \\ 0 & 4. \end{bmatrix} \tag{7.16}$$

and

$$\lambda_R = .001 \tag{7.17}$$

so that

$$R_{MODE} = \frac{B_R}{\lambda_R + m} = \begin{bmatrix} 2. & 0 \\ 0 & 2 \end{bmatrix} . \tag{7.18}$$

Figures 7.13 and 7.14 show the experimental mean square state errors. The curve for the MAP estimator using the inverted Wishart characterization is not shown because it is almost the same as the MAP estimate for the uniform characterization. It is readily apparent that the RF and MAP algorithms are far superior to the SH estimator. Figures 7.15 through 7.17 show the experimental mean square of the difference between the terms in $\hat{R}_{ML}$ and the corresponding terms in the adaptive estimate of R. Again, a great improvement is obtained by the MAP and RF estimators.

G. Summary

The experimental results reported in this chapter seem to indicate that for the first order system:

(1) When R is unknown, the RF and MAP algorithms are superior to the SH estimator when $R_t >> Q$, with

the MAP estimator being superior to the RF algorithm. As Q becomes larger, relative to R_t, the MAP estimator degrades the most while the SH algorithm degrades the least. If P(0/0) is made larger, the MAP algorithm is relatively unaffected while the SH and RF algorithms are significantly degraded.

(2) For R unknown, on-line MAP and RF algorithms can be obtained which are only slightly degraded from their off-line counterparts.

(3) For Q unknown, the MAP estimator seems to be superior to the others when $Q_t >> R$. However, the RF and SH algorithms are about equal. As R becomes larger, relative to Q_t, the MAP estimator degrades in performance. The RF estimator never does seem to do any better than the SH estimator.

(4) For R and Q unknown, the MAP estimator is useless while the RF and SH estimators perform about the same.

For the second order system, with R unknown, the MAP and RF estimators are far superior to the SH estimator.

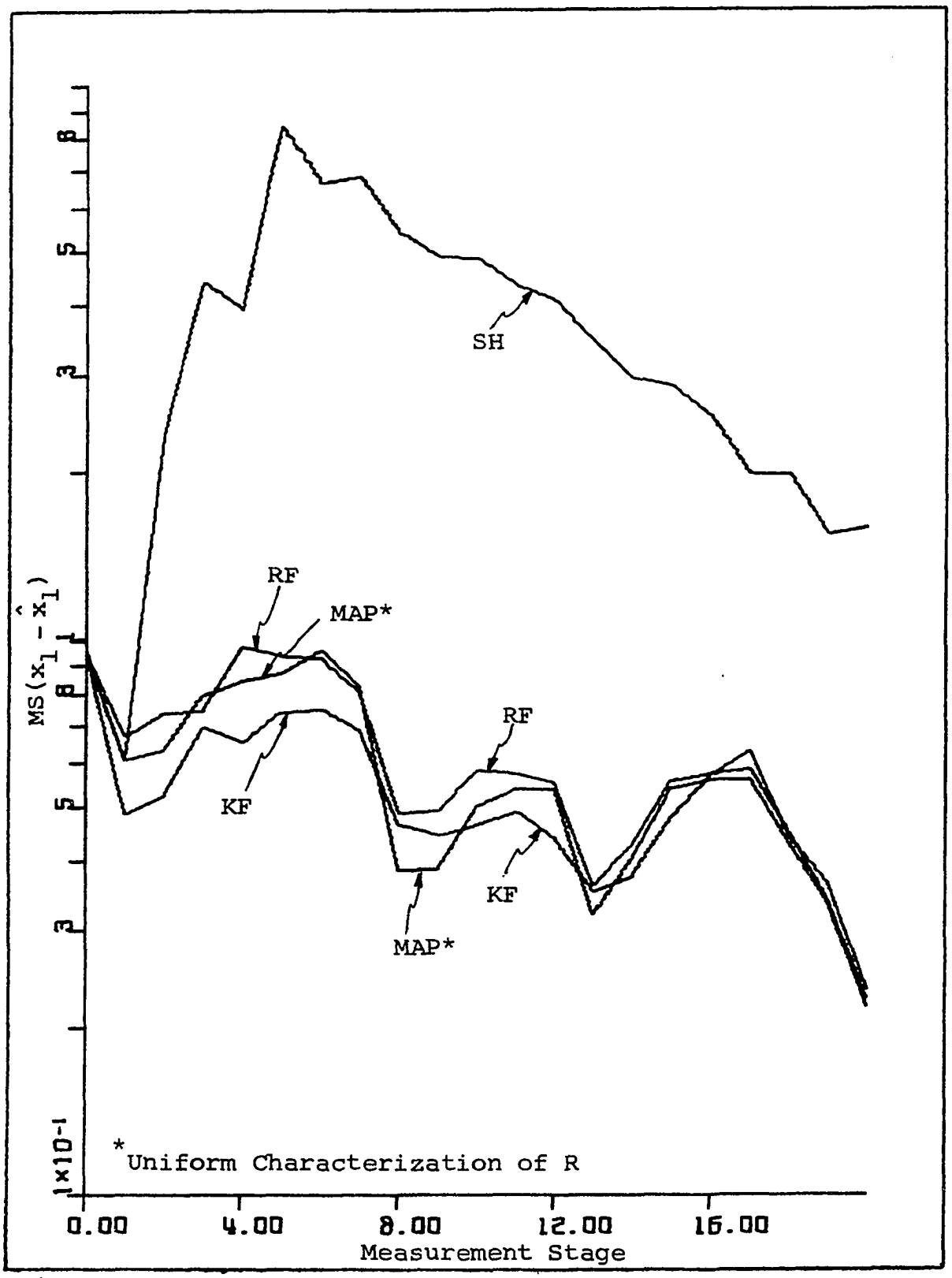

Figure 7.13. Mean square estimation errors for x_1

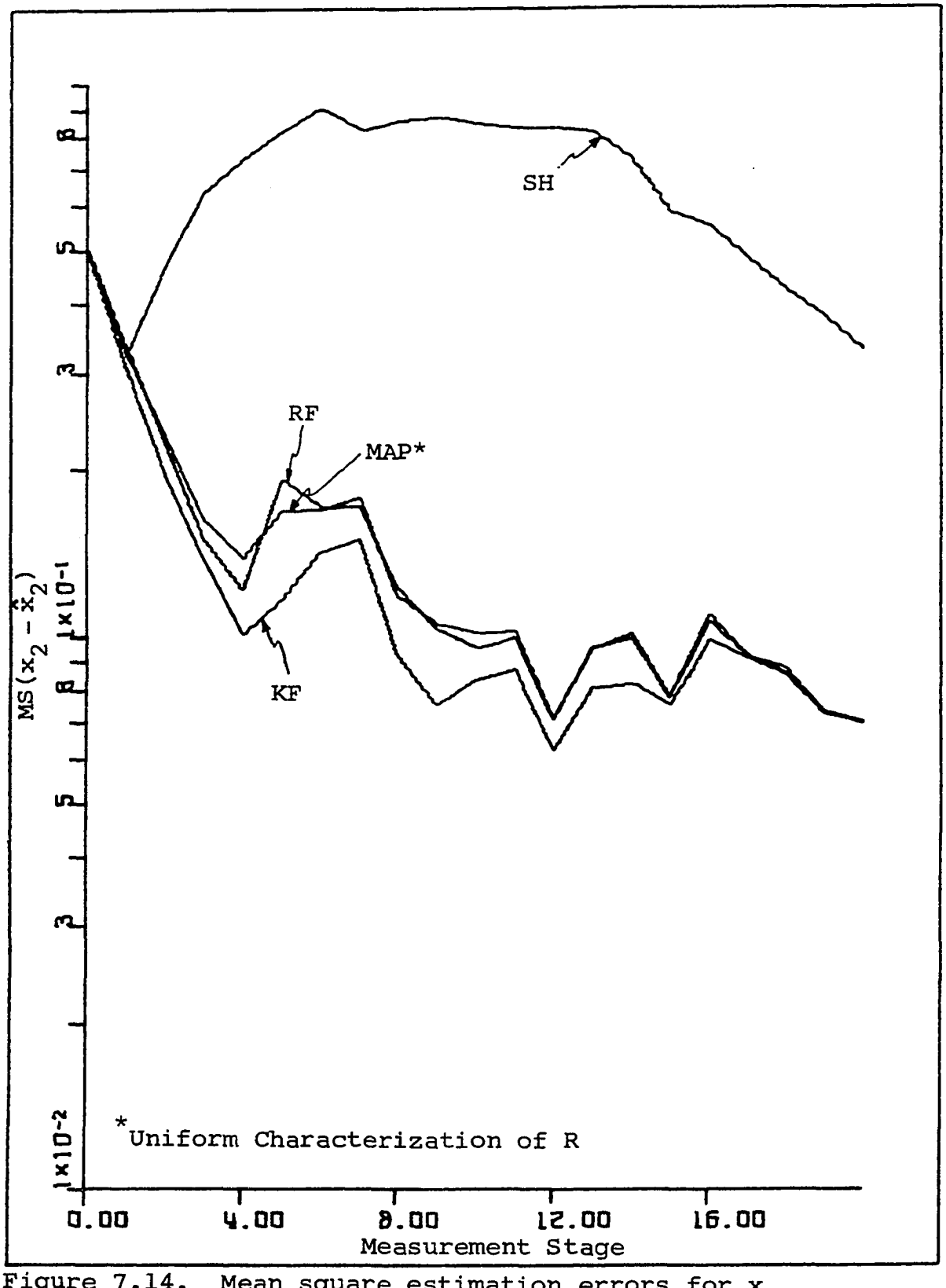

Figure 7.14. Mean square estimation errors for x_2

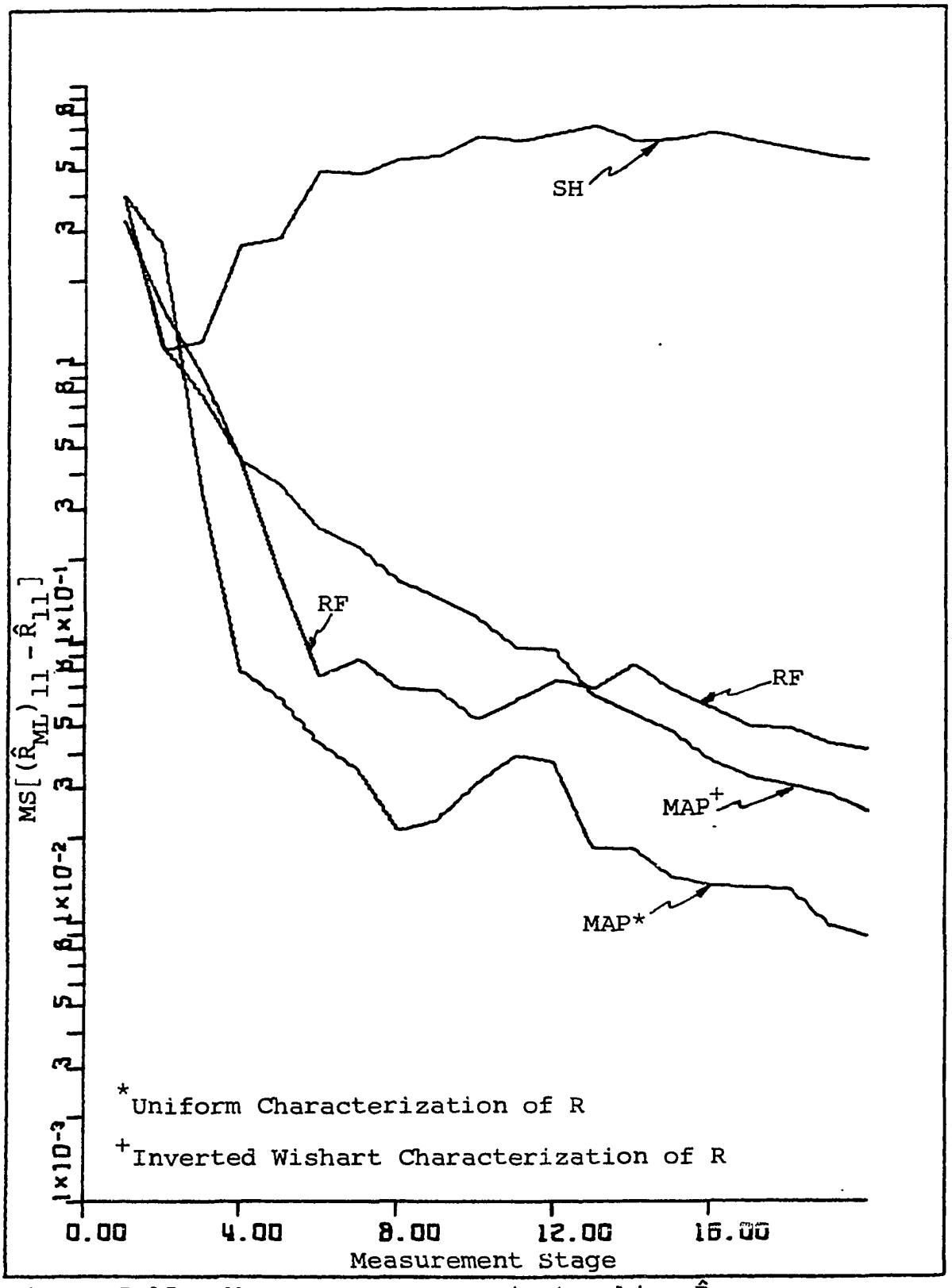

Figure 7.15. Mean square error in tracking $\hat{R}_{ML}$

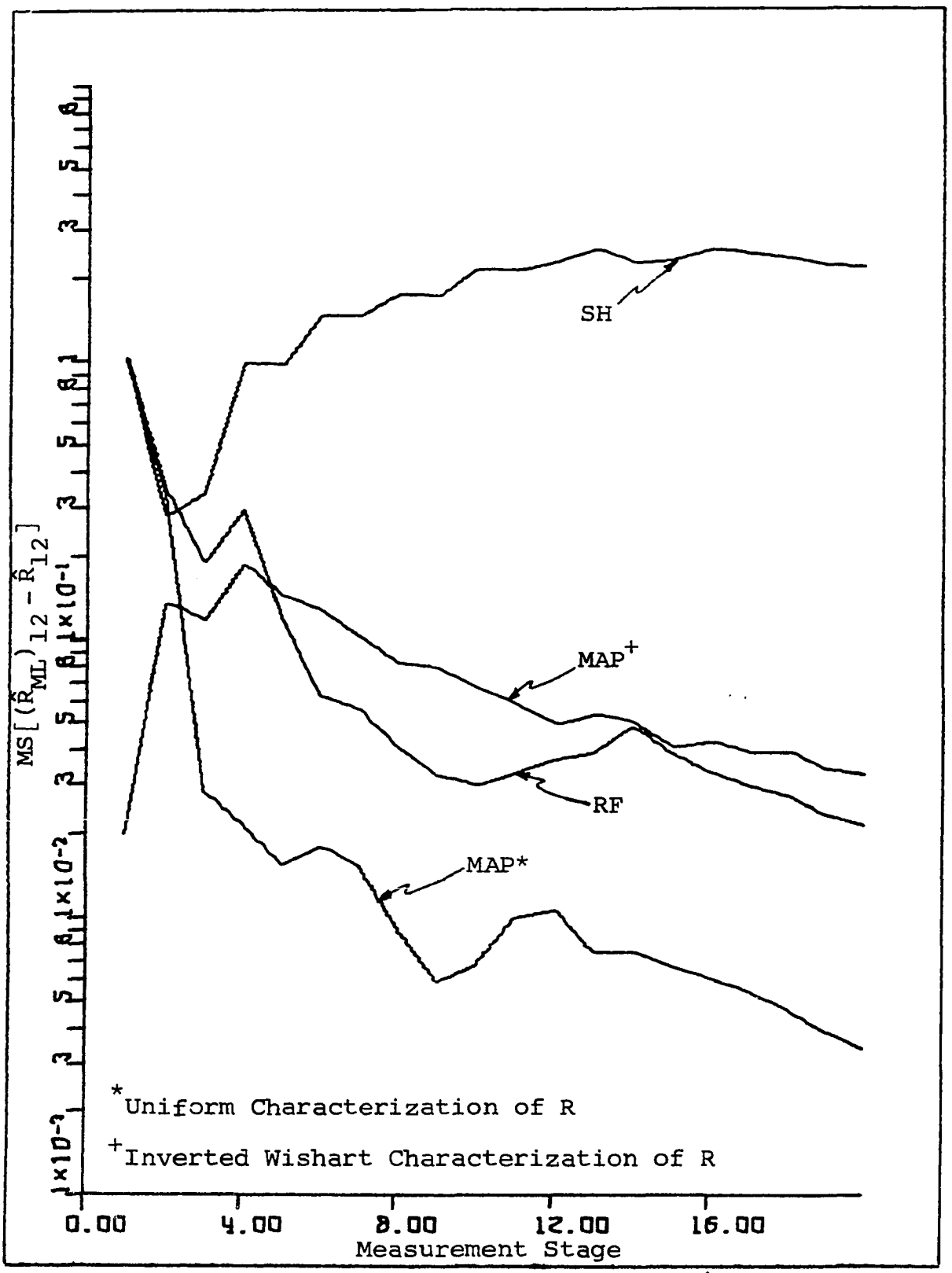

Figure 7.16. Mean square error in tracking $\hat{R}_{ML}$

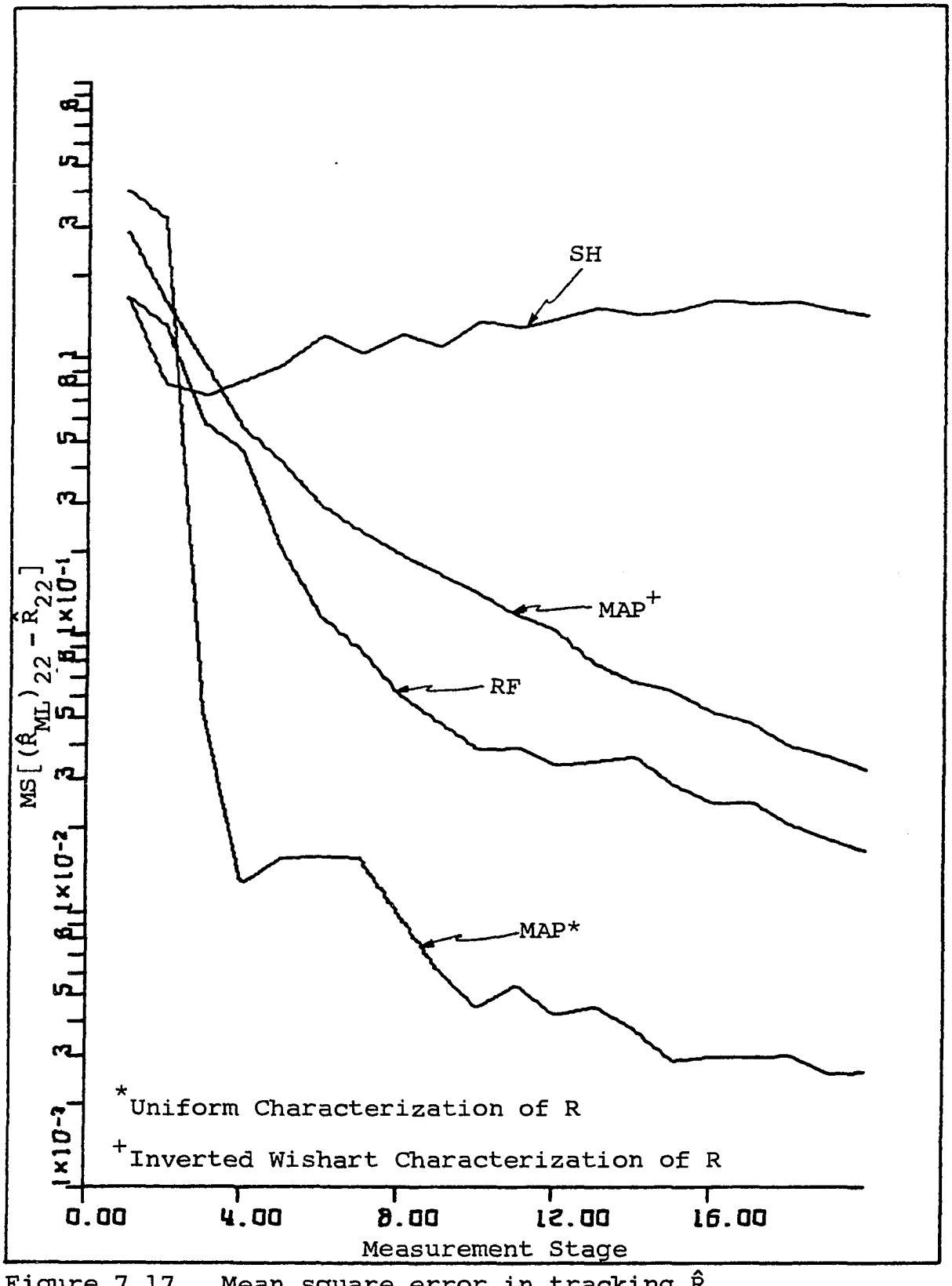

Figure 7.17. Mean square error in tracking $\hat{R}_{ML}$

VIII. CONCLUSIONS AND RECOMMENDATIONS

This dissertation considered the problem of estimating the states of a linear discrete dynamical system when the covariance matrices R and/or Q were unknown. Two new adaptive estimators, called the RF and MAP algorithms, were developed which jointly estimate the state variables and the unknown R and/or Q. The new feature common to both estimators is the use of easily implementable estimators of R and/or Q in a reprocessing configuration with the Kalman-filter algorithm. Under certain conditions the reprocessing of measurements enables these adaptive estimators to quickly "bootstrap" themselves close to the optimal (Kalman-filter with the true R and Q) configuration. This occurs when: (1) only R unknown with relatively "noisy" measurements, (2) only Q unknown with relatively "noiseless" measurements (only MAP algorithm). In these cases, the rate and accuracy of convergence seems to be significantly supeior to the adaptive estimator of Sage and Husa (5). Furthermore, it seems that slightly degraded on-line versions of the MAP and RF algorithms are possible.

In addition, it may also be concluded that under the above conditions the MAP and RF algorithm may be suited for off-line identification of R or Q, especially when only "short" sample sequences are available.

Obviously, not all interesting aspects of this type of problem have been answered by this dissertation.

Pertinent and useful extensions would be:

(1) Unknown Q and/or R that are slowly varying: It seems likely that the on-line MAP or RF algorithms could be extended to this case by reprocessing and using only the last N_B measurements in the covariance estimators; or by using some kind of exponential weighting of the past reprocessed measurement residuals.

(2) Making the MAP estimator of R and Q unbiased when the optimal measurement residuals are available (see Equation 7.5). This might further improve the convergence properties of the MAP estimator.

(3) Investigate the solution of the MAP TPBVP by discrete invariant imbedding techniques. This would yield a nonlinear recursive estimator that would have great computational advantages (if it worked) over the present MAP algorithm.

(4) Extend the MAP approach to dynamical systems described by differential equations with continuous measurements. In this case an a posteriori probability density functional would have to be minimized.

(5) Apply the intuitive approach of Chapter IV to the continuous adaptive filter of Sage and Husa (5).

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X. ACKNOWLEDGMENTS

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To his wife, Meridee, no acknowledgment can possibly convey the appreciation for her sacrifices and understanding over the past years.

XI. APPENDIX A

This appendix will show the derivation of Equations 4.19, 4.20, and 4.21. It was shown in Chapter IV that, from Equation 4.17, the average value of the R estimator in the reprocessor was

$$E[\hat{R}(N)] = R_t + \frac{1}{N} \sum_{k=1}^{N} H(k)[P_a(k/k-1) - P_c(k/k-1)]H(k)^t \tag{A.1}$$

where the superscript p and subscript N, indicating a quantity from the p^{th} cycle of the reprocessing at stage N, have been temporarily dropped for conciseness. The "closeness" of $\hat{R}(N)$ to its mean was defined in Equation 4.18 to be the sum of the variances of each element of $\hat{R}(N)$

$$d_r(N) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{m} E\{([\hat{R}(N) - E(\hat{R}(N))]_{ij})^2\} \quad . \tag{A.2}$$

Since for any m x m matrix, A,

$$Tr(AA^t) = \sum_{i=1}^{m} \sum_{j=1}^{m} ([A]_{ij})^2 \tag{A.3}$$

then since $\hat{R}(N) - E(\hat{R}(N))$ is symmetric

$$\begin{aligned} d_r(N) &= E\{ \sum_{i=1}^{m} \sum_{j=1}^{m} ([\hat{R}(N) - E(\hat{R}(N))]_{ij})^2 \} \\ &= E\{Tr([\hat{R}(N) - E(\hat{R}(N))][\hat{R}(N) - E(\hat{R}(N))])\} \\ &= Tr\{E([\hat{R}(N) - E(\hat{R}(N))][\hat{R}(N) - E(\hat{R}(N))])\} \quad . \end{aligned} \tag{A.4}$$

Now from Equation 4.15,

$$\hat{R}(N) = \frac{1}{N} \sum_{k=1}^{N} \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - H(k)P_c(k/k-1)H(k)^t \tag{A.5}$$

where the "computed" covariance, $P_c(k/k-1)$, is used in Equation A.5 to indicate that an estimate of R is being used in the Kalman-filter equations. Therefore,

$$\hat{R}(N) - E(\hat{R}(N)) = \frac{1}{N}\{ \sum_{k=1}^{N} \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - H(k)P_a(k/k-1)H(k)^t\} - R_t \quad . \tag{A.6}$$

Let

$$A(k)^* \triangleq \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - H(k)P_a(k/k-1)H(k)^t - R_t \quad . \tag{A.7}$$

Then

$$\hat{R}(N) - E(\hat{R}(N)) = \frac{1}{N} \sum_{k=1}^{N} A(k)^* \tag{A.8}$$

so

$$E\{ [\hat{R}(N) - E(\hat{R}(N))]^2\} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E\{A(i)^*A(j)^*\} \tag{A.9}$$

now

$$E\{A(i)^*A(j)^*\} = E\{\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t\}$$

$$- E\{\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t[H(j)P_a(j/j-1)H(j)^t+R_t]\}$$

$$- E\{[H(i)P_a(i/i-1)H(i)^t+R_t]\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t\}$$

$$+ E\{[H(i)P_a(i/i-1)H(i)^t + R_t][H(j)P_a(j/j-1)H(j)^t + R_t]\} \quad (A.10)$$

Let

$$Y_a(i,j) \triangleq E\{\tilde{z}(i/i-1)\tilde{z}(j/j-1)^t\} \quad (A.11)$$

then from Equation 4.16

$$\boxed{Y_a(i,i) = H(i)P_a(i/i-1)H(i)^t + R_t} \quad (A.12)$$

so that

$$E\{A(i)^*A(j)^*\} = E\{\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t\}$$

$$- Y_a(i,i)Y_a(j,j) \quad . \quad (A.13)$$

The rp^{th} element of

$$\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t$$

is

$$[\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t]_{rp} = \sum_{k=1}^{m} \mu_{ri}\mu_{ki}\mu_{kj}\mu_{pj} \quad (A.14)$$

where μ_{ri} is the r^{th} element of the column vector $\tilde{z}(i/i-1)$.

Since the random variables μ_{ki} are Gaussian with zero mean,

$$E\{\mu_{ri}\mu_{ki}\mu_{kj}\mu_{pj}\} = E[\mu_{ri}\mu_{ki}]E[\mu_{kj}\mu_{pj}]$$
$$+ E[\mu_{ri}\mu_{kj}]E[\mu_{ki}\mu_{pj}]$$
$$+ E[\mu_{ri}\mu_{pj}]E[\mu_{ki}\mu_{kj}] \qquad \text{(A.15)}$$

so that

$$E[\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t]_{rp}$$
$$= \sum_{k=1}^{m} E[\mu_{ri}\mu_{ki}]E[\mu_{kj}\mu_{pj}]$$
$$+ \sum_{k=1}^{m} E[\mu_{ri}\mu_{kj}]E[\mu_{ki}\mu_{pj}]$$
$$+ \sum_{k=1}^{m} E[\mu_{ri}\mu_{pj}]E[\mu_{ki}\mu_{kj}]$$
$$= \sum_{k=1}^{m} [Y_a(i,i)]_{rk}[Y_a(j,j)]_{kp}$$
$$+ \sum_{k=1}^{m} [Y_a(i,j)]_{rk}[Y_a(i,j)]_{kp}$$
$$+ \sum_{k=1}^{m} [Y_a(i,j)]_{rp}[Y_a(i,j)]_{kk} \quad . \qquad \text{(A.16)}$$

Therefore,

$$E[\tilde{z}(i/i-1)\tilde{z}(i/i-1)^t\tilde{z}(j/j-1)\tilde{z}(j/j-1)^t]$$
$$= Y_a(i,i)Y_a(j,j)+Y_a(i,j)Y_a(i,j)+Y_a(i,j)\mathrm{Tr}(Y_a(i,j)). \qquad \text{(A.17)}$$

From Equation A.13

$$E\{A(i)^*A(j)^*\} = Y_a(i,j)Y_a(i,j) + Y_a(i,j)Tr(Y_a(i,j)) \tag{A.18}$$

and from Equation A.9 and A.18

$$E\{[\hat{R}(N) - E[\hat{R}(N)]]^2\} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} Y_a(i,j)^2 + Y_a(i,j)Tr(Y_a(i,j)) \quad . \tag{A.19}$$

So from A.19 and A.4

$$\boxed{d_r(N) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} Tr(Y_a(i,j)^2) + [Tr(Y_a(i,j))]^2} \quad . \tag{A.20}$$

This is the desired Equation 4.19.

$Y_a(i,i)$ has already been computed in Equation A.12. For $Y_a(i,j)$, the resulting expression will be considerably more complicated.

From Equations 2.2 and 4.2,

$$\begin{aligned}\tilde{z}(k/k-1) &= z(k) - H(k)\hat{x}_a(k/k-1) \\ &= H(k)[x(k) - \hat{x}_a(k/k-1)] + v(k) \\ &= H(k)\tilde{x}_a(k/k-1) + v(k)\end{aligned} \tag{A.21}$$

where $\tilde{x}_a(k/k-1)$ is the "actual" a priori estimation error of a Kalman-filter using an incorrect value for R. Assuming $j > i$, from Equations A.11 and A.21,

$$Y_a(i,j) = E[\tilde{z}(i/i-1)\tilde{z}(j/j-1)^t]$$

$$= E[H(i)\tilde{x}_a(i/i-1)\tilde{x}_a(j/j-1)^t H(j)^t]$$

$$+ E[H(i)\tilde{x}_a(i/i-1)v(j)^t] + E[v(i)\tilde{x}_a(j/j-1^t H(j)^t] \quad (0)$$

$$+ E[v(i)v(j)^t] \quad (0)$$

$$= H(i)E[\tilde{x}_a(i/i-1)\tilde{x}_a(j/j-1)^t]H(j)^t$$

$$+ E[v(i)\tilde{x}_a(j/j-1)^t]H(j)^t \qquad (A.22)$$

where from Equation 4.3, $\tilde{x}_a(i/i-1)$ is a function of $x(0)$, $w(0), \ldots, w(i-1), v(1), \ldots, v(i-1)$. By definition, from Chapter II, $v(j)$ is uncorrelated with these random vectors, hence, uncorrelated with $\tilde{x}_a(i/i-1)$. Also, by definition from Chapter II, $v(i)$ is uncorrelated with $v(j)$ for all $i \neq j$. From Equations 2.1, 4.1, 4.3, and A.21,

$$\tilde{x}_a(j/j-1) \triangleq x(j) - \hat{x}_a(j/j-1) = A(j,j-1)x(j-1) + G(j-1)w(j-1)$$

$$- A(j,j-1)\hat{x}_a(j-1/j-1)$$

$$= A(j,j-1)x(j-1) + G(j-1)w(j-1) - A(j,j-1)\{\hat{x}_a(j-1/j-2)$$

$$+ K_c(j-1)[H(j-1)\tilde{x}_a(j-1/j-2) + v(j-1)]\}$$

$$= A(j,j-1)[I - K_c(j-1)H(j-1)]\tilde{x}_a(j-1/j-2)$$

$$- A(j,j-1)K_c(j-1)v(j-1) + G(j-1)w(j-1) \qquad (A.23)$$

where again the subscript "a" is used to denote the actual estimate of its error and the subscript "c" is used to indicate that the gain, $K_c(j)$, is a computed gain based on an estimate of R, and is not the optimal gain based on knowledge of the true R.

Successively re-substituting Equation A.23 into itself,

$$\tilde{x}_a(j/j-1) = \prod_{e=1}^{j-i} A(j-e+1,j-e)[I - K_c(j-e)H(j-e)]\tilde{x}_a(i/i-1)$$

$$+ \sum_{m=1}^{j-i} \left\{ \prod_{e=1}^{m-1} A(j-e+1,j-e)[I - K_c(j-e)H(j-e)] \right\}$$

$$\cdot [G(j-m)w(j-m) - A(j-m+1,j-m)K_c(j-m)v(j-m)] \qquad (A.24)$$

where

$$\prod_{e=1}^{o} \triangleq I \quad .$$

Let

$$T(i,j) \triangleq \prod_{e=1}^{j-i} A(j-e+1,j-e)[I - K_c(j-e)H(j-e)] \quad . \qquad (A.25)$$

Using Equations A.24 and A.25, for $j > i$,

$$E[\tilde{x}_a(i/i-1)\tilde{x}_a(j/j-1)^t] = E[\tilde{x}_a(i/i-1)\tilde{x}_a(i/i-1)^t]T(i,j)^t$$

$$+ \sum_{m=1}^{j-i} \{E[\tilde{x}_a(i/i-1)w(j-m)^t]G(j-m)^t$$

$$-E[\tilde{x}_a(i/i-1)v(j-m)^t]K_c(j-m)^tA(j-m+1,j-m)^t\}T(j+1-m,j)^t$$

$$= P_a(i/i-1)T(i,j)^t \quad . \tag{A.26}$$

Also for $j > i$,

$$E[v(i)\tilde{x}_a(j/j-1)^t] = E[v(i)\tilde{x}_a(i/i-1)^t]T(i,j)^t$$

$$+ \sum_{m=1}^{j-i}\{E[v(i)w(j-m)^t]G(j-m)^t$$

$$- E[v(i)v(j-m)^t]K_c(j-m)^tA(j-m+1,j-m)^t\}T(j+1-m,j)^t$$

$$= -R_tK_c(i)^tA(i+1,i)^tT(i+1,j)^t \quad . \tag{A.27}$$

Therefore, from Equations A.22, A.26 and A.27, for $j > i$

$$Y_a(i,j) = H(i)P_a(i/i-1)T(i,j)^tH(j)^t$$

$$- R_tK_c(i)^tA(i+1,i)^tT(i+1,j)^tH(j)^t$$

$$= \{H(i)P_a(i/i-1)[I - K_c(i)H(i)]^t$$

$$- R_tK_c(i)^t\}A(i+1,i)^tT(i+1,j)^tH(j)^t \tag{A.28}$$

Now from Equation 4.5,

$$K_c(i) = P_c(i/i-1)H(i)^t[H(i)P_c(i/i-1)H(i)^t + \hat{R}(N)^*]^{-1} \tag{A.29}$$

where $\hat{R}(N)^*$ is some best past estimate of R. This yields

$$\hat{R}(N)^{*}K_c(i)^t = H(i)P_c(i/i-1)[I - K_c(i)H(i)]^t \quad . \qquad (A.30)$$

Also from Equation A.30,

$$-R_t K_c(i)^t = -R_t[\hat{R}(N)^{*}]^{-1}H(i)P_c(i/i-1)[I - K_c(i)H(i)]^t \quad . \qquad (A.31)$$

Substituting Equation A.31 into A.28

$$Y_a(i,j) = \{H(i)P_a(i/i-1) - R_t[\hat{R}(N)^{*}]^{-1}H(i)P_c(i/i-1)\}$$

$$\cdot[I - K_c(i)H(i)]^t A(i+1,i)^t T(i+1,j)^t H(j)^t$$

So for $j > i$, using Equation A.25

$$\boxed{Y_a(i,j) = \{H(i)P_a(i/i-1) - R_t[\hat{R}(N)^{*}]^{-1}H(i)P_c(i/i-1)\} \cdot T(i,j)^t H(j)^t} \qquad (A.32)$$

Now from Equation 4.6,

$$P_c(i/i) = [I - K_c(i)H(i)]P_c(i/i-1)$$

so that Equation A.25 can be rewritten as

$$\boxed{T(i,j) = \prod_{e=1}^{j-i} A(j-e+1,j-e)P_c(j-e/j-e)P_c(j-e/j-e-1)^{-1}} \qquad (A.33)$$

Equations A.32 and A.33 are then part of Equations 4.20 and 4.21 for $j > i$.

Now for $i > j$, from Equations A.11 and A.21,

$$Y_a(i,j) = E[\tilde{z}(i/i-1)\tilde{z}(j/j-1)^t]$$

$$= E[H(i)\tilde{x}_a(i/i-1)\tilde{x}_a(j/j-1)^t H(j)^t]$$

$$+ E[H(i)\tilde{x}_a(i/i-1)v(j)^t] + E[v(i)\tilde{x}_a(j/j-1)^t H(j)^t]$$

$$+ E[v(i)v(j)^t]$$

$$= H(i)E[\tilde{x}_a(i/i-1)\tilde{x}_a(j/j-1)^t]H(j)^t$$

$$+ H(i)E[\tilde{x}_a(i/i-1)v(j)^t] \quad . \qquad \text{(A.34)}$$

Examination of Equation A.34 shows that its transpose is the same as Equation A.22 except that the stage variables i and j are interchanged. Therefore, for $i > j$, Equation A.32 is transposed and i and j are interchanged yielding,

$$\boxed{Y_a(i,j) = H(i)T(j,i)\{P_a(j/j-1)H(j)^t - P_c(j/j-1)H(j)^t[\hat{R}(N)^*]^{-1}R_t\}} \quad . \qquad \text{(A.35)}$$

Equations A.12, A.32, A.33, and A.35 comprise Equations 4.20 and 4.21, with the superscript, p, and subscript, N, dropped for conciseness.

XII. APPENDIX B

Equations 4.22 and 4.24 will be derived in this appendix. The development that results in Equation 4.22 will be shown first. From Equation 4.19, $d_r(N)$ can be rewritten as

$$d_r(N) = \frac{1}{N^2}\{\sum_{i=1}^{N} Tr[Y_a(i,i)^2] + [Tr(Y_a(i,i))]^2\}$$

$$+ \frac{2}{N^2}\{\sum_{i=2}^{N}[\sum_{j=1}^{i-1} Tr[Y_a(i,j)^2] + [Tr(Y_a(i,j))]^2]\} \quad (B.1)$$

where $Y_a(i,j)$ is given by Equations 4.20 and 4.21, and where, again, the subscript, N, and the superscript, p, have been dropped for conciseness.

Using the assumptions of uniform complete observability and uniform complete controllability, Price (20) shows that the homogeneous equation

$$y(k) = P_c(k/k)[P_c(k/k-1)]^{-1}A(k,k-1)y(k-1) \quad (B.2)$$

is uniformly asymptotically stable in the large. This implies that

$$\lim_{i\to\infty}[\prod_{e=j}^{i}[P_c(e/e)[P_c(e/e-1)]^{-1}A(e,e-1)]^t]^t = 0 \quad . \quad (B.3)$$

By removing the first two terms, $P_c(i/i)[P_c(i/i-1)]^{-1}$, and the last term, $A(j,j-1)$, and since this does not affect the convergence of remaining product,

$$\lim_{i\to\infty}\Big[\prod_{e=j}^{i-1}[A(e+1,e)P_c(e/e)[P_c(e/e-1)]^{-1}]^t\Big]^t = 0 \qquad (B.4)$$

or equivalently from Equation A.33,

$$\lim_{(i-j)\to\infty} T(j,i) = 0 \quad . \qquad (B.5)$$

Now due to the boundedness of $P_a(k/k-1)$ and $P_c(k/k-1)$, the assumed invertibility of $\hat{R}(N)_N^{p-1}$, and the result in Equation B.5, then $Y_a(i,j)$ approaches the null matrix for $i >> j$ which intern yields

$$\lim_{(i-j)\to\infty} Tr(Y_a(i,j)^2) + [Tr(Y_a(i,j))]^2 = 0 \quad . \qquad (B.6)$$

Equation B.6 is a sufficient condition for the Cesáro mean to converge to 0 (25). Therefore,

$$\lim_{i\to\infty} \frac{1}{i-1} \sum_{j=1}^{i-1} Tr(Y_a(i,j)^2) + [Tr(Y_a(i,j))]^2 = 0 \quad . \qquad (B.7)$$

Let

$$C_{i-1} \triangleq \frac{1}{i-1} \sum_{j=1}^{i-1} Tr(Y_a(i,j)^2)+[Tr(Y_a(i,j))]^2 \qquad (B.8)$$

then Equation B.1 can be rewritten as

$$d_r(N) = \frac{1}{N^2} \sum_{i=1}^{N} Tr(Y_a(i,i)^2)+[Tr(Y_a(i,i))]^2+\frac{2}{N^2} \sum_{i=2}^{N} (i-1)C_{i-1} \; . \qquad (B.9)$$

Now for any $N > M > 0$, (similar to proof in Parzen (26, p. 75)),

$$\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} = \frac{2}{N^2}\sum_{i=2}^{M}(i-1)C_{i-1} + \frac{2}{N^2}\sum_{i=M+1}^{N}(i-1)C_{i-1} \quad . \text{(B.10)}$$

Now

$$\frac{2}{N^2}\sum_{i=M+1}^{N}(i-1)C_{i-1} \leq \frac{2}{N}\sum_{i=M+1}^{N}C_{i-1} \leq \frac{2}{N}\sum_{i=M+1}^{N}\sup_{M<k}(C_k) = \frac{2(N-M)}{N}\sup_{M<k}(C_k). \quad \text{(B.11)}$$

So

$$\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} \leq \frac{2}{N^2}\sum_{i=2}^{M}(i-1)C_{i-1} + 2\sup_{M<k}(C_k) \quad . \quad \text{(B.12)}$$

Therefore, keeping M fixed and letting N tend to ∞,

$$\lim_{N\to\infty}\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} \leq 0 + 2\sup_{M<k}(C_k) \quad . \quad \text{(B.13)}$$

Now letting M tend to ∞ and using Equations B.7 and B.8 in Equation B.13,

$$\lim_{M\to\infty}\lim_{N\to\infty}\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} = \lim_{N\to\infty}\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} \leq 0 \quad \text{(B.14)}$$

which implies that (all terms are positive)

$$\lim_{N\to\infty}\frac{2}{N^2}\sum_{i=2}^{N}(i-1)C_{i-1} = 0 \quad . \quad \text{(B.15)}$$

Now due to the boundedness of $P_a(k/k-1)$, then from Equation 4.20, $Y_a(i,i)$ is bounded, so that the terms $Tr(Y_a(i,i)^2) + [Tr(Y_a(i,i))]^2$ are bounded by some large

number, M. Thus,

$$\frac{1}{N^2}\sum_{i=2}^{N} \mathrm{Tr}(Y_a(i,i)^2) + [\mathrm{Tr}(Y_a(i,i))]^2 \leq \frac{1}{N^2}\sum_{i=2}^{N} M \qquad \text{(B.16)}$$

so

$$\lim_{N\to\infty} \frac{1}{N^2}\sum_{i=2}^{N} \mathrm{Tr}(Y_a(i,i)^2) + [\mathrm{Tr}(Y_a(i,i)]^2 = 0 \quad . \qquad \text{(B.17)}$$

Therefore, by Equations B.9, B.15 and B.17,

$$\boxed{\lim_{N\to\infty} d_r(N) = 0} \qquad \text{(B.18)}$$

which is the desired Equation 4.22.

It will now be shown that for a stationary first order system in the form of Equation 2.1 and 2.2, the successive reprocessing scheme will always converge under the assumptions needed to obtain Equation B.18.

Since the system is stationary, it is assumed that for large N, the Kalman-filter and the system in Equations 2.1 2.2 are in steady state. Therefore, in the steady state, for large N,

$$\hat{R}(N)=E[\hat{R}(N)]= R_t + \frac{1}{N}\sum_{k=1}^{N} H[P_{ass} - P_{css}]H^t$$

$$= R_t - HE_{cass}H^t \qquad \text{(B.19)}$$

where the stage variables have been dropped to indicate stationary and steady state values and where

$$E_{ca}(k) \triangleq P_c(k/k-1) - P_a(k/k-1) \quad . \tag{B.20}$$

Nishamura shows that for an assumed true R, R_t, and an estimated R, $\hat{R}(N)^*$, then (6)

$$E_{ca}(k+1) = .$$

$$A(k,k-1)[I - K_c(k)H(k)]E_{ca}(k)[I - K_c(k)H(k)]^t A(k,k-1)^t$$

$$+ A(k,k-1)K_c(k)[\hat{R}(N)^* - R_t]K_c(k)^t A(k,k-1)^t \tag{B.21}$$

For the scalar stationary case and in the steady state,

$$E_{cass} = A^2[I - K_{css}H]^2 E_{cass} + A^2 K_{css}^2(\hat{R}(N)^* - R_t) \tag{B.22}$$

where the terms are now scalars. E_{cass} is then

$$E_{cass} = \frac{A^2 K_{css}^2(\hat{R}(N)^* - R_t)}{1 - A^2(1-K_{css}H)^2} \quad . \tag{B.23}$$

From Equation 4.5, for this steady state scalar case,

$$K_{css} = \frac{P_{css}H}{H^2 P_{css} + \hat{R}(N)^*} \quad . \tag{B.24}$$

Substituting Equation B.24 into B.23 and the result into Equation B.19,

$$\hat{R}(N) - R_t = \frac{-A^2(\hat{R}(N)^* - R_t)}{1 + \frac{2\hat{R}(N)^*}{H^2 P_{css}} + \frac{(\hat{R}(N)^*)^2(1-A^2)}{H^4 P_{css}^2}} \quad . \tag{B.25}$$

From the stationary and stability assumptions, $A^2 < 1$, so that for $\hat{R}(N)^* > 0$ and $\hat{R}(N)^* \neq R_t$,

$$|\hat{R}(N) - R_t| < |\hat{R}(N)^* - R_t| \tag{B.26}$$

indicating that at the end of each reprocessing, the re-estimated value of R, $\hat{R}(N)$, will be closer to R_t than the previous one, $\hat{R}(N)^*$.

XIII. APPENDIX C

This appendix will derive the expressions for the mean and dispersion of the GQG^t estimator when only Q is unknown, and when both R and Q are unknown. The mean and dispersion of the R estimator when both R and Q are unknown will also be developed.

First it is assumed that R is known and Q is unknown. The nonrecursive form of Equation 4.8 is

$$G\hat{Q}(N)G^t = \frac{1}{N}\sum_{k=1}^{N}\{K_c(k)\tilde{z}(k/k-1)\tilde{z}(k/k-1)^t K_c(k)^t + P_c(k/k) - A(k,k-1)P_c(k-1/k-1)A(k,k-1)^t\} \qquad (C.1)$$

where, as before, the subscript N and superscript p have been omitted for conciseness. Also, $K_c(k)$ indicates the "computed" Kalman gain using only an estimate of GQG^t. Using Equation 4.16, the mean value of $G\hat{Q}(N)G^t$ becomes

$$E\{G\hat{Q}(N)G^t\} = \frac{1}{N}\sum_{k=1}^{N}\{K_c(k)[H(k)P_a(k/k-1)H(k)^t + R_t]K_c(k)^t + P_c(k/k) - A(k,k-1)P_c(k-1/k-1)A(k,k-1)^t\}. \qquad (C.2)$$

Using Equations 4.4, 4.5 (with $\hat{R}(N)_N^{p-1} = R_t$), and 4.6 in Equation C.2,

$$E\{G\hat{Q}(N)G^t\} = G\hat{Q}(N)^*G^t + \frac{1}{N}\sum_{k=1}^{N} K_c(k)H(k)[P_a(k/k-1) - P_c(k/k-1)]H(k)^t K_c(k)^t \tag{C.3}$$

where $\hat{Q}(N)^*$ is the estimate of Q used in the Kalman-filter algorithm. Thus, if the true value of Q is used in the Kalman-filter, then $P_a(k/k-1) = P_c(k/k-1)$ and $G\hat{Q}(N)^*G^t = GQ_tG^t$ so that by Equation C.3, the estimator of GQG^t is unbiased. However, in general, it will not be unbiased.

As before, the "closeness" of $G\hat{Q}(N)G^t$ to its mean is defined

$$d_q(N) \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} E\{([G\hat{Q}(N)G^t - E(G\hat{Q}(N)G^t)]_{ij})^2\} \tag{C.4}$$

$$= \mathrm{Tr}\{E[(G\hat{Q}(N)G^t - E(G\hat{Q}(N)G^t))^2]\} \quad . \tag{C.5}$$

Now from Equations C.1 and C.2,

$$G\hat{Q}(N)G^t - E(G\hat{Q}(N)G^t) = \frac{1}{N}\sum_{k=1}^{N}\{K_c(k)[\tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - H(k)P_a(k/k-1)H(k)^t - R_t]K_c(k)^t\} \tag{C.6}$$

so that

$$E[(G\hat{Q}(N)G^t - E(G\hat{Q}(N)G^t))^2] = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{e=1}^{N} K_c(k)S(k,e)K_c(e)^t \tag{C.7}$$

where from Equation A.12,

$$S(k,e) \triangleq E\{[\tilde{z}(k/k-1)\tilde{z}(k/k-1)^t - Y_a(k,k)]K_c(k)^t K_c(e)[\tilde{z}(e/e-1)\tilde{z}(e/e-1)^t - Y_a(e,e)]\} \tag{C.8}$$

$$= E\{\tilde{z}(k/k-1)\tilde{z}(k/k-1)^t K_c(k)^t K_c(e)\tilde{z}(e/e-1)\tilde{z}(e/e-1)^t - \tilde{z}(k/k-1)\tilde{z}(k/k-1)^t K_c(k)^t K_c(e) Y_a(e,e) - Y_a(k,k) K_c(k)^t K_c(e)\tilde{z}(e/e-1)\tilde{z}(e/e-1)^t + Y_a(k,k)K_c(k)^t K_c(e) Y_a(e,e)\} \quad . \tag{C.9}$$

Now

$$[S(k,e)]_{ij} = E\{\sum_p \sum_q [\mu_{ik}\mu_{pk}[K_c(k)^t K_c(e)]_{pq}\mu_{qe}\mu_{je} - \mu_{ik}\mu_{pk}[K_c(k)^t K_c(e)]_{pq} Y_a(e,e)_{qj} - Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq}\mu_{pe}\mu_{je} + Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq} Y_a(e,e)_{qj}]\} \tag{C.10}$$

where, as before, μ_{ik} is the i^{th} element of the column vector $\tilde{z}(k/k-1)$. Since the random variables μ_{ik} are Gaussian with zero mean,

$$E\{\mu_{ik}\mu_{pk}\mu_{qe}\mu_{je}\} = E[\mu_{ik}\mu_{pk}]E[\mu_{qe}\mu_{je}]$$

$$+ E[\mu_{ik}\mu_{qe}]E[\mu_{pk}\mu_{je}]$$

$$+ E[\mu_{ik}\mu_{je}]E[\mu_{pk}\mu_{qe}] \quad . \qquad \text{(C.11)}$$

By the definition in Equation A.11, Equation C.11 becomes

$$E\{\mu_{ik}\mu_{pk}\mu_{qe}\mu_{je}\} = Y_a(k,k)_{ip}Y_a(e,e)_{qj}$$

$$+ Y_a(k,e)_{iq}Y_a(k,e)_{pj}$$

$$+ Y_a(k,e)_{ij}Y_a(k,e)_{pq} \qquad \text{(C.12)}$$

so that

$$[S(k,e)]_{ij} = \sum_p \sum_q [Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq}Y_a(e,e)_{qj}$$

$$+ Y_a(k,e)_{iq}Y_a(k,e)_{pj}[K_c(k)^t K_c(e)]_{pq}$$

$$+ Y_a(k,e)_{ij}Y_a(k,e)_{pq}[K_c(k)^t K_c(e)]_{pq}$$

$$- Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq}Y_a(e,e)_{qj}$$

$$- Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq}Y_a(e,e)_{qj}$$

$$+ Y_a(k,k)_{ip}[K_c(k)^t K_c(e)]_{pq}Y_a(e,e)_{qj}] \qquad \text{(C.13)}$$

$$= \sum_p \sum_q [Y_a(k,e)_{iq}[K_c(e)^t K_c(k)]_{qp}Y_a(k,e)_{pj}$$

$$+ Y_a(k,e)_{ij}Y_a(k,e)_{pq}[K_c(e)^t K_c(k)]_{qp}] \quad . \qquad \text{(C.14)}$$

Then

$$S(k,e) = Y_a(k,e)K_c(e)^t K_c(k)Y_a(k,e)$$

$$+ Y_a(k,e)\mathrm{Tr}[Y_a(k,e)K_c(e)^t K_c(k)] \quad . \qquad (C.15)$$

From Equations C.15, C.7, and C.5,

$$d_q(N) = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{e=1}^{N}\mathrm{Tr}\{K_c(k)[Y_a(k,e)K_c(e)^t K_c(k)Y_a(k,e)$$

$$+ Y_a(k,e)\mathrm{Tr}(Y_a(k,e)K_c(e)^t K_c(k))]K_c(e)^t\} \qquad (C.16)$$

or

$$d_q(N) = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{e=1}^{N}\mathrm{Tr}[K_c(k)Y_a(k,e)K_c(e)^t K_c(k)Y_a(k,e)K_c(e)^t]$$

$$+ [\mathrm{Tr}(K_c(k)Y_a(k,e)K_c(e)^t)]^2 \quad . \qquad (C.17)$$

Since R_t is known, then $Y_a(i,j)$, as defined in Equation A.11, will be slightly different. In this case, Equation A.30 will be changed to

$$R_t K_c(i)^t = H(i)P_c(i/i-1)[I - K_c(i)H(i)]^t \qquad (C.18)$$

so that by substituting Equation C.18 into A.28, the end result is

$$Y_a(i,j) = \begin{cases} H(i)P_a(i/i-1)H(i)^t + R_t & i=j \\ H(i)[P_a(i/i-1) - P_c(i/i-1)]T(i,j)^t H(j)^t & j>i \\ H(i)T(j,i)[P_a(j/j-1) - P_c(j/j-1)]H(j)^t & i>j \end{cases} \qquad (C.19)$$

Now if the assumption of uniform complete controllability and observability hold, then

$$\lim_{N\to\infty} d_q(N) = 0 \tag{C.20}$$

by a proof almost entirely identical to the one stablishing Equation B.18. Thus, under the above assumption, the estimator of GQG^t will converge, in a mean square sense, to its mean. Since it is generally biased, $G\hat{Q}(N)G^t$ will not converge to the true value after one cycle of the reprocessor. To experimentally determine whether successive reprocessing will converge to the true value for large N, the equation

$$G\hat{Q}(N)G^t = E(G\hat{Q}(N)G^t) \tag{C.21}$$

can be used for large N in a similar fashion as was explained for Equation 4.23.

It will now be shown that for a stationary first order system in the form of Equations 2.1 and 2.2, the successive reprocessing scheme will always converge under the assumptions needed to obtain Equation C.20. This development is similar to the one leading to Equation B.26. For large N, the system and the Kalman-filter are assumed to be in the steady state so that from Equations C.21 and C.3

$$G\hat{Q}(N)G^t = G\hat{Q}(N)*G^t + \frac{1}{N}\sum_{k=1}^{N} K_{css}H[P_{ass}-P_{css}]H^tK_{css}^t \tag{C.22}$$

where the stage variables have been dropped to indicate stationary and steady state values. Using Equation B.20 and since all the quantities in Equation C.22 are scalar, then

$$G^2\hat{Q}(N) = G^2\hat{Q}(N)^* - K_{css}^2 H^2 E_{cass} \quad . \tag{C.23}$$

From Nishimura (6), it can be shown that if R is known, and an estimate $G^2\hat{Q}(N)^*$ is used in the Kalman-filter algorithm of the reprocessor, then

$$E_{cass} = \frac{G^2\hat{Q}(N)^* - G^2Q_t}{[1-A^2(1-K_{css}H)^2]} \tag{C.24}$$

where

$$K_{css} = \frac{P_{css}H}{H^2P_{css} + R} \tag{C.25}$$

then using Equations C.23, C.24, and C.25, it can be shown that

$$G^2\hat{Q}(N) - G^2Q_t = \frac{G^2\hat{Q}(N)^* - G^2Q_t}{1 + \dfrac{H^4P_{css}^2}{2H^2P_{css}R + R^2(1-A^2)}} \quad . \tag{C.26}$$

Since $A < 1$ and $R > 0$, then when $G^2\hat{Q}(N)^* \neq G^2Q_t$,

$$|G^2\hat{Q}(N) - G^2Q_t| < |G^2\hat{Q}(N)^* - G^2Q_t| \tag{C.27}$$

indicating that at the end of each cycle of reprocessing,

the re-estimated value of G^2Q, $G^2\hat{Q}(N)$, will be closer to G^2Q_t than the previous one, $G^2\hat{Q}(N)^*$.

Now, finally, the convergence properties of the R and GQG^t estimators will be discussed when both are unknown. In this case the expressions for the mean and dispersion of $\hat{R}(N)$ are identical to Equations 4.17 and 4.19, respectively. Consequently, the estimate, $\hat{R}(N)$, converges to its mean under the same sufficient conditions as mentioned before.

The expression for the mean of $G\hat{Q}(N)G^t$ is different than in Equation C.3, due to the added uncertainty in R. Using Equations 4.4, 4.5, and 4.6 in Equation C.2,

$$E[G\hat{Q}(N)G^t] = G\hat{Q}(N)^*G^t + \frac{1}{N}\sum_{k=1}^{N} K_c(k)H(k)[P_a(k/k-1) - P_c(k/k-1)]H(k)^tK_c(k)^t + K_c(k)[R_t - \hat{R}(N)^*]K_c(k)^t \quad . \tag{C.28}$$

However, the expression for the dispersion of $G\hat{Q}(N)G^t$ is the same as Equation C.17, but with $Y_a(i,j)$ given by Equation 4.20 and 4.21. The convergence of $G\hat{Q}(N)G^t$ to its mean as N becomes large still holds under the aforementioned sufficient conditions.

For the scalar stationary case, with both R and Q unknown, the convergence of the successive reprocessing to the true values of GQG^t and R has not yet been proven.

XIV. APPDNEIX D

This appendix will first show an example of the multivariate uniform density shown in Equation 5.16. The density functions for P[Q] and P[R] are then derived by assuming R^{-1} and Q^{-1} are characterized by Wishart distributions. An example for a scalar R is then shown.

Let R be a 2 x 2 matrix characterized by the multivariate density,

$$P[R] = \begin{cases} C_R & R \in E_R^* \\ 0 & \text{otherwise} \end{cases} \qquad (D.1)$$

E_R^* could be specified by placing upper and lower bounds on the diagonal elements and requiring the off-diagonal element to be such that R is always positive definite. In other words,

$$E_R^* = \left\{ (r_{11}, r_{12}, r_{22}) : r_{11min} \le r_{11} \le r_{11max}, r_{22min} \le r_{22} \le r_{22max}, r_{12} < \sqrt{r_{11} r_{22}} \right\} \qquad (D.2)$$

In this case,

$$C_R = \frac{9}{8(r_{11max}^{3/2} - r_{11min}^{3/2})(r_{22max}^{3/2} - r_{22min}^{3/2})} \; . \qquad (D.3)$$

Now let B_R be a known, real, symmetric, and positive definite matrix of order m. Let C be a random, real,

symmetric matrix of order m. Let $(c_{11}, c_{12}, \ldots, c_{mm})$ be the vector of m(m-1)/2 elements of C with

$$E_c = \{(c_{11}, c_{12}, \ldots, c_{mm}) : C \text{ is positive definite}\} \quad (D.4)$$

then a form of the Wishart density for C is (27),

$$P[C] = \begin{cases} \frac{|C|^{\lambda_R - 1}}{K_R} \exp\{-\mathrm{Tr}(B_R C)\} ; (c_{11}, c_{12}, \ldots, c_{mm}) \varepsilon E_c \\ 0 \qquad\qquad \text{otherwise} \end{cases} \quad (D.5)$$

where

$$K_R = \pi^{\frac{m(m-1)}{4}} \Gamma(\lambda_R)\Gamma(\lambda_R + 1/2) \ \ldots \ \Gamma(\lambda_R + \tfrac{m-1}{2}) |B_R|^{-\lambda_R - \frac{m-1}{2}} \quad (D.6)$$

and

$$\lambda_R > 0 \quad . \quad (D.7)$$

Now let $R^{-1} = C$ and define

$$E_R = \{(r_{11}, r_{12}, \ldots, r_{mm}) : R \text{ is positive definite}\}. \quad (D.8)$$

Therefore, $R = C^{-1}$ defines a one to one transformation on C to allow

$$P[R] = P[C = R^{-1}]|J| \quad (D.9)$$

where

$$|J| \triangleq \left| \det \frac{\partial(c_{11}, c_{12}, \ldots, c_{mm})}{\partial(r_{11}, r_{12}, \ldots, r_{mm})} \right| \quad . \quad (D.10)$$

Since C^{-1} is positive definite if and only if C is positive definite, then $(r_{11},\ldots,r_{mm})\varepsilon E_R$ if and only if $(c_{11},\ldots,c_{mm})\varepsilon E_C$.

Therefore,

$$P[C=R^{-1}] = \begin{cases} \dfrac{1}{K_R|R|\ R^{-1}}\exp\{-\mathrm{Tr}(B_R R^{-1})\} : (r_{11},\ldots,r_{mm})\varepsilon E_R \\ \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{D.11}$$

From Deemer and Olkin (28),

$$|J| = \frac{1}{|R|^{m+1}} \quad . \tag{D.12}$$

So that from Equations D.12, D.11 and D.9,

$$P[R] = \begin{cases} \dfrac{1}{K_R|R|^{\lambda_R+m}}\exp\{-\mathrm{Tr}(B_R R^{-1})\} & R \text{ is positive definite} \\ \\ 0 & \text{otherwise} \end{cases} \tag{D.13}$$

By setting the matrix gradient (29) of $-\ln P[R]$ to zero, the mode is found to be

$$R_{max} = \frac{B_R}{\lambda_R+m} \quad . \tag{D.14}$$

A similar development made for Q resulted in

$$P[\Omega] = \begin{cases} \frac{1}{K_\Omega |\Omega|^{\lambda_\Omega + r}} \exp\{-\mathrm{Tr}(B_\Omega \Omega^{-1})\} ; \Omega \text{ is positive definite} \\ 0 \qquad \text{otherwise} \end{cases} \tag{D.15}$$

where K_Ω is a normalizing scalar, similar to Equation D.6 and λ_Ω is a positive scalar.

From Equation D.10, it can be seen for a scalar R, (B_R is a scalar)

$$P[R] = \begin{cases} \frac{1}{K_R \, R^{\lambda_R + 1}} \exp(-B_R/R) & R > 0 \\ 0 & R \leq 0 \end{cases} \tag{D.16}$$

which is a form of the inverted gamma density function. This density is graphed in Figure D.1 for various values of λ_R.

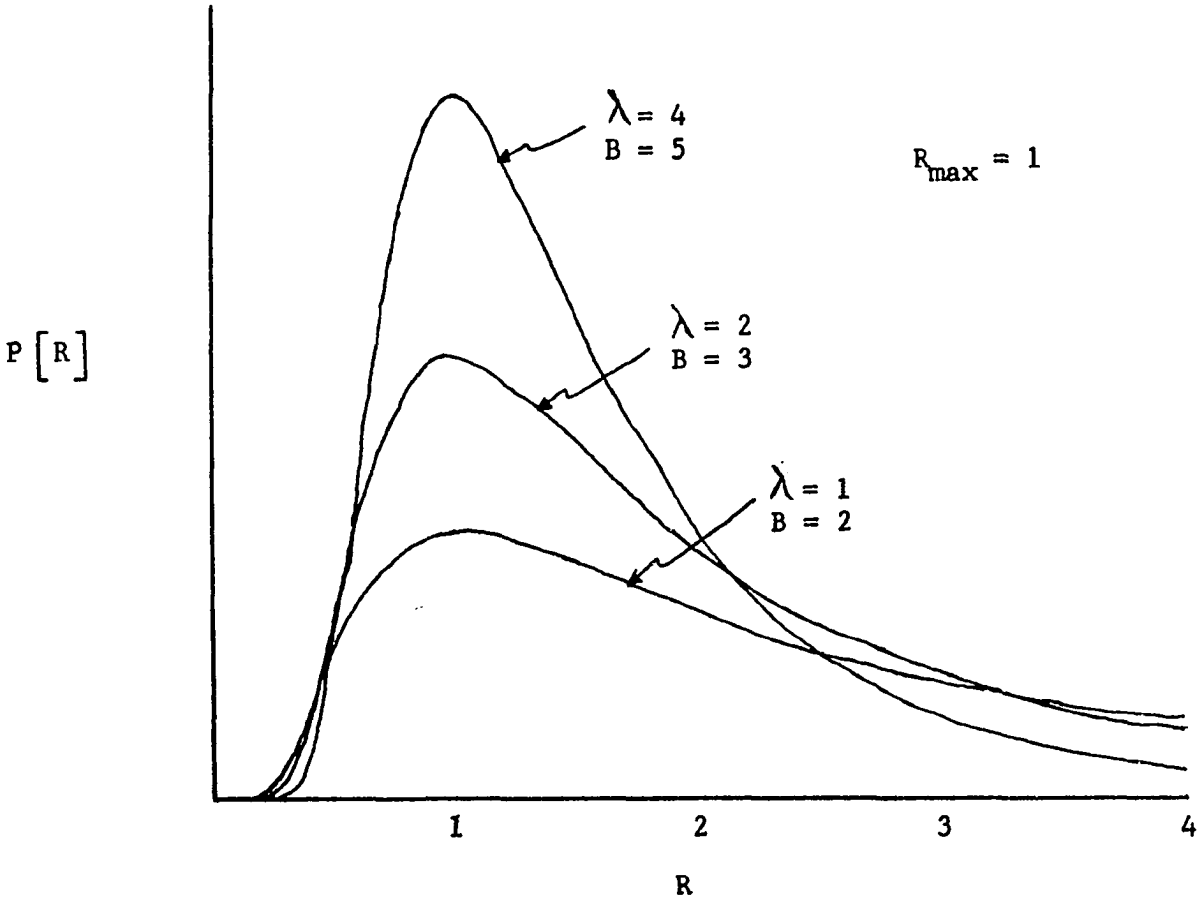

Figure D.1. Inverted gamma density function

XV. APPENDIX E

In this appendix the necessary conditions for a local extremum will be determined for the case where R and Q are characterized by multivariate uniform densities and when R and Q are characterized by the inverted Wishart densities.

From Equations 6.7 and 6.10 it is desired to minimize

$$I^{**} = \frac{k}{2}\ln|R| + \frac{1}{2}\sum_{j=1}^{k}\ln|G(j-1)QG(j-1)^t|$$

$$+ \frac{1}{2}[\|x(0)\|^2_{P(0/0)^{-1}} + \sum_{j=1}^{k}\|z(j)-H(j)x(j)\|^2_{R^{-1}}$$

$$+ \sum_{j=1}^{k}\|w(j-1)\|^2_{Q^{-1}}] + \sum_{j=1}^{k} s(j-1)^t[x(j) - A(j,j-1)x(j-1)$$

$$- G(j-1)w(j-1)] \qquad (E.1)$$

with respect to $\{X(k),Q,R,W(k-1),S(k-1)\}$. The necessary conditions for a local extremum can be found by setting the partial derivatives of I**, with respect to each unknown, to zero.

The partial derivative of the scalar J(a) with respect to a vector, a, is defined to be

$$\frac{\partial J(a)}{\partial a} \triangleq \left[\frac{\partial J(a)}{\partial a_1} \cdots \frac{\partial J(a)}{\partial a_n}\right]^t$$

where a_j is the j^{th} component of the vector a. Then, using the gradient identities in Sage (24), for R and Q symmetric,

$$\left.\frac{\partial I^{**}}{\partial x(0)}\right|_{\hat{}} = P(0/0)^{-1}\hat{x}(0/k) - A(1,0)^{t}s(0/k) = 0^{1}$$

or

$$\boxed{\hat{x}(0/k) = P(0/0)A(1,0)^{t}s(0/k)} \qquad \text{(E.2)}$$

$$\left.\frac{\partial I^{**}}{\partial x(j)}\right|_{\hat{}} = -H(j)^{t}\hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)] + s(j-1/k)$$

$$-A(j+1,j)^{t}s(j/k) = 0 \qquad j = 1,\ldots,k-1$$

or

$$s(j-1/k) = A(j+1,j)^{t}s(j/k) + H(j)^{t}\hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)]$$

$$j = 1,\ldots,k-1$$

(E.3)

$$\left.\frac{\partial I^{**}}{\partial x(k)}\right|_{\hat{}} = H(k)^{t}\hat{R}(k)^{-1}[z(k) - H(k)\hat{x}(k/k)] + s(k-1/k) = 0$$

or

$$s(k-1/k) = H(k)^{t}\hat{R}(k)^{-1}[z(k) - H(k)\hat{x}(k/k)] \quad . \qquad \text{(E.4)}$$

From Equation E.4, it can be seen that $s(k/k) = 0$ in order that Equation E.3 hold for $j = k$. This is called the end-point boundary condition on $s(j/k)$. Therefore, Equations E.3 and E.4 are equivalent to

---

[1] $B\big|_{\hat{}}$ indicates that the quantity, B, is evaluated at the extremum values $\hat{x}(0/k),\ldots,\hat{x}(k/k),\hat{R}(k),\hat{Q}(k)$, etc. ...

$$\boxed{s(j-1/k) = A(j+1,j)^t s(j/k) + H(j)^t \hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)] \qquad j = 1,\ldots,k}$$

(E.5)

$$\boxed{s(k/k) = 0}$$ (E.6)

$$\left.\frac{\partial I^{**}}{\partial s(j)}\right|_{\hat{}} = \hat{x}(j+1/k) - A(j+1,j)\hat{x}(j/k) - G(j)\hat{w}(j/k) = 0$$

or

$$\boxed{\hat{x}(j+1/k) = A(j+1,j)\hat{x}(j/k) + G(j)\hat{w}(j/k) \qquad j = 0,\ldots,k-1}$$

(E.7)

$$\left.\frac{\partial I^{**}}{\partial w(j)}\right|_{\hat{}} = \hat{Q}(k)^{-1}\hat{w}(j/k) - G(j)^t s(j/k) = 0 \qquad j = 0,\ldots,k-1$$

or

$$\boxed{\hat{w}(j/k) = \hat{Q}(k)G(j)^t s(j/k) \qquad j = 0,1,\ldots,k-1}$$

(E.8)

Now from Athans' development (29) on gradient matrices,

$$\left.\frac{\partial I^{**}}{\partial Q}\right|_{\hat{}} = \frac{1}{2}\sum_{j=1}^{k}[\hat{Q}(k)^{-1}]^t - \frac{1}{2}\sum_{j=1}^{k}[\hat{Q}(k)^{-1}\hat{w}(j-1/k)\hat{w}(j-1/k)^t\hat{Q}(k)^{-1}]^t$$

$$= 0$$

or

$$\boxed{\hat{Q}(k) = \frac{1}{k} \sum_{j=1}^{k} \hat{w}(j-1/k)\hat{w}(j-1/k)^t} \tag{E.9}$$

$$\left.\frac{\partial I^{**}}{\partial R}\right|_{\hat{}} = \frac{k}{2}[\hat{R}(k)^{-1}]^t - \frac{1}{2} \sum_{j=1}^{k} [\hat{R}(k)^{-1}[z(j)-H(j)\hat{x}(j/k)]$$

$$[z(j) - H(j)\hat{x}(j/k)]^t \hat{R}(k)^{-1}]^t = 0$$

or

$$\boxed{\hat{R}(k) = \frac{1}{k} \sum_{j=1}^{k} [z(j) - H(j)\hat{x}(j/k)][z(j) - H(j)\hat{x}(j/k)]^t} \tag{E.10}$$

Equations E.2, E.5, E.6, E.7, E.8, E.9, and E.10 are the necessary conditions for a local extremum, corresponding to Equations 6.12 through 6.18.

Now the necessary conditions will be derived for the case where R and Q are characterized by the inverted Wishart densities. The cost function, corresponding to this case, is given in Equation 6.22. Attaching the equality constraints of Equation 6.2 to the cost function by vector Lagrange multipliers, the resulting cost function is related to Equation E.1 by

$$I^{++} = I^{**} + (\lambda_R+m)\ln|R| + \mathrm{Tr}(B_R R^{-1}) + (\lambda_Q+r)\ln|Q| + \mathrm{Tr}(B_Q Q^{-1}). \tag{E.11}$$

Thus the gradients with respect to $\{X(k),W(k-1),S(k-1)\}$ will be the same as before, yielding Equations E.2, E.5, E.6, E.7, and E.8. However,

$$\left.\frac{\partial I^{++}}{\partial Q}\right|_{\hat{}} = \left.\frac{\partial I^{**}}{\partial Q}\right|_{\hat{}} + (\lambda_Q+r)[\hat{Q}(k)^{-1}]^t - [\hat{Q}(k)^{-1}B_Q\hat{Q}(k)^{-1}]^t = 0$$

or

$$\boxed{\hat{Q}(k) = \frac{1}{k+2(\lambda_Q+r)}\{\sum_{j=1}^{k}\hat{w}(j-1/k)\hat{w}(j-1/k)^t + 2B_Q\}} \qquad \text{(E.12)}$$

$$\left.\frac{\partial I^{++}}{\partial R}\right|_{\hat{}} = \left.\frac{\partial I^{**}}{\partial R}\right|_{\hat{}} + (\lambda_R+m)[\hat{R}(k)^{-1}]^t - [\hat{R}(k)^{-1}B_R\hat{R}(k)^{-1}]^t$$

or

$$\boxed{\hat{R}(k) = \frac{1}{k+2(\lambda_R+m)}\{\sum_{j=1}^{k}[z(j)-H(j)\hat{x}(j/k)][z(j)-H(j)\hat{x}(j/k)]^t+2B_R\}}$$

(E.13)

Equations E.12 and E.13 correspond to Equations 6.24 and 6.25.

XVI. APPENDIX F

Necessary conditions for an extremum of $P[X(k)/Z(k)]$ are derived for the inverted Wishart characterization of R and Q. It is desired to maximize $P[X(k)/Z(k)]$ with respect to X(k). Its Bayesian equivalent is

$$P[X(k)/Z(K)] = \int_{E_Q} \cdots \int \int_{E_R} \cdots \int P[X(k),Q,R/Z(k)] \prod_{ij}^{m} dr_{ij} \prod_{ij}^{m} dq_{ij} \tag{F.1}$$

where E_R and E_Q are the appropriate regions in the m(m-1)/2 and r(r-1)/2 Euclidean spaces of the $(r_{11}, r_{12}, \ldots, r_{mm})$ and $(q_{11}, q_{12}, \ldots, q_{rr})$ vectors upon which P[Q] and P[R] are defined.

First, R and Q will be characterized by the inverted Wishart densities defined in Equations D.13 and D.15. Substituting Equations D.13, D.15, and 5.10 into Equation 5.9 and recognizing Equation 5.2 and 5.3, then Equation F.1 becomes

$$P[X(k)/Z(k)] =$$

$$\left[\exp -\tfrac{1}{2}\left[\|x(0)\|^2_{P(0/0)^{-1}} + \sum_{j=1}^{k} \|v(j)\|^2_{R^{-1}} + \|w(j-1)\|^2_{Q^{-1}}\right]\right]$$

$$\int_{E_Q} \cdots \int \int_{E_R} \cdots \int \frac{\cdot \exp -\left[\mathrm{Tr}(B_R R^{-1}) + \mathrm{Tr}(B_Q Q^{-1})\right]}{\left[P[Z(k)]C|R|^{\frac{k}{2}} \prod_{j=1}^{k} |G(j-1)QG(j-1)^t|\right]^{1/2}} \prod_{ij}^{m} dr_{ij} \prod_{ij}^{r} dq_{ij}$$

$$K_Q |Q|^{\lambda_Q + r} \; K_R |R|^{\lambda_R + m} \Big] \tag{F.2}$$

$$E_R = \{(r_{11}, r_{12}, \ldots, r_{mm}) : R \text{ is p.d.}\} \tag{F.3}$$

$$E_Q = \{(q_{11}, q_{i2}, \ldots, q_{rr}) : Q \text{ is p.d.}\} \tag{F.4}$$

$$v(j) = z(j) - H(j)x(j) \qquad j = 1, \ldots, k \tag{F.5}$$

$$x(j) = A(j, j-1)x(j-1) + G(j-1)w(j-1) \qquad j = 1, \ldots, k \tag{F.6}$$

Since G(j-1) is invertible, then

$$|G(j-1)QG(j-1)^t|^{1/2} = |G(j-1)|\,|Q|^{1/2} \tag{F.7}$$

let

$$C^* \triangleq P[Z(k)] \prod_{j=1}^{k} |G(j-1)| K_R K_Q \tag{F.8}$$

and notice that

$$\sum_{j=1}^{k} \|v(j)\|^2_{R^{-1}} = \sum_{j=1}^{k} \mathrm{Tr}(v(j)v(j)^t R^{-1})$$

$$= \mathrm{Tr}\Big[\Big[\sum_{j=1}^{k} v(j)v(j)^t\Big]R^{-1}\Big] \quad . \tag{F.9}$$

Then Equation F.2 becomes,

$$P[X(k)/Z(k)] = \tag{F.10}$$

$$\Big(\exp\{-.5\|x(0)\|^2_{P(0/0)^{-1}} - \mathrm{Tr}[[.5 \sum_{j=1}^{k} v(j)v(j)^t + B_R]R^{-1}]\}$$

$$\int\cdots\int_{E_Q} \int\cdots\int_{E_R} \frac{\cdot \exp\{-\mathrm{Tr}[[.5 \sum_{j=1}^{k} w(j-1)w(j-1)^t + B_Q]Q^{-1}]\}\Big)}{C^* |R|^{\lambda_R + \frac{k}{2} + m} |Q|^{\lambda_Q + \frac{k}{2} + r}} \prod_{ij}^{m} dr_{ij} \prod_{ij}^{r} dq_{ij}$$

now let

$$B_R^* \triangleq .5 \sum_{j=1}^{k} v(j)v(j)^t + B_R \tag{F.11}$$

$$B_Q^* \triangleq .5 \sum_{j=1}^{k} w(j-1)w(j-1)^t + B_Q \tag{F.12}$$

$$\lambda_R^* \triangleq \lambda_R + \frac{k}{2} \tag{F.13}$$

$$\lambda_Q^* \triangleq \lambda_Q + \frac{k}{2} \tag{F.14}$$

$$K_R^* \triangleq \pi^{\frac{m(m-1)}{4}} \Gamma(\lambda_R^*)\Gamma(\lambda_R^* + \tfrac{1}{2}) \ldots \Gamma(\lambda_R^* + \tfrac{m-1}{2}) |B_R^*|^{-\lambda_R^* - \frac{m-1}{2}} \tag{F.15}$$

$$K_Q^* \triangleq \pi^{\frac{r(r-1)}{4}} \Gamma(\lambda_Q^*)\Gamma(\lambda_Q^* + \tfrac{1}{2}) \ldots \Gamma(\lambda_Q^* + \tfrac{r-1}{2}) |B_Q^*|^{-\lambda_Q^* - \frac{r-1}{2}} \tag{F.16}$$

Then Equation F.10 becomes

$$P[X(k)/Z(K)] =$$

$$\int_{E_Q} \cdots \int \int_{E_R} \cdots \int \frac{\exp[-.5||x(0)||^2 P(0/0)^{-1} - \mathrm{Tr}(B_R^* R^{-1}) - \mathrm{Tr}(B_Q^* Q^{-1})]}{C^* |R|^{\lambda_R^* + m} |Q|^{\lambda_Q^* + r}} \prod_{ij}^{m} dr_{ij} \prod_{ij}^{r} dq_{ij} \tag{F.17}$$

Now since $\lambda_R^* > 0$ and B_R^* is p.d., then from Equation D.13,

$$P[R] = \begin{cases} \dfrac{\exp(-\mathrm{Tr}(B_R^* R^{-1}))}{K_R^* |R|^{\lambda_R^* + m}} & ; \quad R \in E_R \\ 0 & \text{otherwise} \end{cases} \qquad \text{(F.18)}$$

so that

$$\int_{E_R} \cdots \int \frac{\exp(-\mathrm{Tr}(B_R^* R^{-1}))}{|R|^{\lambda_R^* + m}} \prod_{ij}^{m} dr_{ij} = K_R^* \quad . \qquad \text{(F.19)}$$

Similarly,

$$\int_{E_Q} \cdots \int \frac{\exp(-\mathrm{Tr}(B_Q^* Q^{-1}))}{|Q|^{\lambda_Q^* + r}} \prod_{ij}^{r} dq_{ij} = K_Q^* \quad . \qquad \text{(F.20)}$$

Therefore, using Equations F.19 and F.20 in Equation F.17,

$$P[X(k)/Z(k)] = \frac{\exp(-.5||x(0)||^2 P(0/0)^{-1}) K_R^* K_Q^*}{C^*} \quad . \qquad \text{(F.21)}$$

Let

$$C^{**} = \frac{C^*}{\pi^{\frac{m(m-1)}{2}} \Gamma(\lambda_R^*) \cdots \Gamma(\lambda_R^* + \frac{m-1}{2}) \pi^{\frac{r(r-1)}{2}} \Gamma(\lambda_Q^*) \cdots \Gamma(\lambda_Q^* + \frac{r-1}{2})} \cdot$$

(F.22)

Substituting Equations F.15, F.16, and F.22 into Equation F.21,

$$P[X(k)/Z(k)] = \frac{\exp(-.5||x(0)||^2 P(0/0)^{-1})}{C^{**} |B_R^*|^{\lambda_R^* + \frac{m-1}{2}} |B_Q^*|^{\lambda_Q^* + \frac{r-1}{2}}} \cdot \qquad \text{(F.23)}$$

From Equations F.5, F.11, F.12

$$I = -\ln C^{**} P[X(k)/Z(k)] = .5\|x(0)\|^2 P(0/0)^{-1} + (\lambda_Q^* + \frac{r-1}{2})$$

$$+ \ln\left|.5 \sum_{j=1}^{k} w(j-1)w(j-1)^t + 2B_Q\right| + (\lambda_R^* + \frac{m-1}{2})$$

$$+ \ln\left|.5 \sum_{j=1}^{k} [z(j)-H(j)x(j)][z(j)-H(j)x(j)]^t + 2B_R\right|$$

(F.24)

Therefore, maximizing $P[X(k)/Z(k)]$ with respect to $X(k)$ is equivalent to minimizing I with respect to $\{X(k),W(k-1)\}$, subject to the equality constraints of Equation F.6. These constraints are attached to I by Lagrange multiplier vectors to get I*,

$$I^* = I + \sum_{j=1}^{k} s(j-1)^t[x(j) - A(j,j-1)x(j-1) - G(j-1)w(j-1)].$$

(F.25)

The objective is to minimize I* with respect to $\{X(k),W(k-1),S(k-1)\}$. It will be shown later that

$$\frac{\partial \ln\left|.5 \sum_{p=1}^{k} [z(p) - H(p(x(p)][z(p) - H(p)x(p)]^t + B\right|}{\partial x(n)}$$

$$= -H(n)^t[.5 \sum_{p=1}^{k} [z(p)-H(p)x(p)][z(p)-H(p)x(p)]^t + B]^{-1}[z(n)-H(n)x(n)].$$

(F.26)

Let

$$\hat{R}(k) \triangleq \frac{1}{\lambda_R^* + \frac{m-1}{2}}\{.5 \sum_{p=1}^{k} [z(p) - H(p)\hat{x}(p/k)][z(p) - H(p)\hat{x}(p/k)]^t + B_R\} \tag{F.27}$$

and

$$\hat{Q}(k) \triangleq \frac{1}{\lambda_Q^* + \frac{r-1}{2}}\{.5 \sum_{p=1}^{k} \hat{w}(p-1/k)\hat{w}(p-1/k)^t + B_Q\} \tag{F.28}$$

then

$$\left.\frac{\partial I^*}{\partial x(0)}\right|_{\hat{}} = P(0/0)^{-1}\hat{x}(0/k) - A(1,0)^t s(0/k) = 0$$

or

$$\boxed{\hat{x}(0/k) = P(0/0)A(1,0)^t s(0/k)} \tag{F.29}$$

$$\left.\frac{\partial I^*}{\partial x(j)}\right|_{\hat{}} = -H(j)^t\hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)] + s(j-1/k)$$
$$-A(j+1,j)^t s(j/k) = 0 \qquad j = 1,\ldots,k-1$$

or

$$s(j-1/k) = A(j+1,j)^t s(j/k) + H(j)^t\hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)]$$
$$j = 1,\ldots,k-1 \tag{F.30}$$

$$\left.\frac{\partial I^*}{\partial x(k)}\right|_{\hat{}} = -H(k)^t\hat{R}(k)^{-1}[z(k) - H(k)\hat{x}(k/k)] + s(k-1/k) = 0$$

or

$$s(k-1/k) = H(k)^t\hat{R}(k)^{-1}[z(k) - H(k)\hat{x}(k/k)] \quad . \tag{F.31}$$

Similarly, as before, Equations F.30 and F.31 can be combined to get

$$\boxed{s(j-1/k) = A(j+1,j)^t s(j/k) - H(j)^t \hat{R}(k)^{-1}[z(j) - H(j)\hat{x}(j/k)]} \qquad j = 1,\ldots,k \quad \text{(F.32)}$$

$$\boxed{s(k/k) = 0} \qquad \text{(F.33)}$$

$$\left.\frac{\partial I^*}{\partial s(j)}\right|_{\hat{}} = \hat{x}(j+1/k) - A(j+1,j)\hat{x}(j/k) - G(j)\hat{w}(j/k) = 0 \qquad j = 0,\ldots,k-1$$

or

$$\boxed{\hat{x}(j+1/k) = A(j+1,j)\hat{x}(j/k) + G(j)\hat{w}(j/k) \qquad j = 0,\ldots,k-1}\ . \qquad \text{(F.34)}$$

Now from Equation F.26, with $z(p) = 0$ for all $p = 1,\ldots,k$, and $H(p) = I$ for all $p = 1,\ldots,k$,

$$\frac{\partial \ln\left|.5 \sum_{p=1}^{k} x(p)x(p)^t + B\right|}{\partial x(n)} = \left[.5 \sum_{p=1}^{k} x(p)x(p)^t + B\right]^{-1} x(n) \qquad \text{(F.35)}$$

so that

$$\left.\frac{\partial I^*}{\partial w(j)}\right|_{\hat{}} = \hat{Q}(k)^{-1}\hat{w}(j/k) - G(j)^t s(j/k) = 0 \qquad j = 0,\ldots,k-1$$

or

$$\hat{w}(j/k) = \hat{Q}(k)G(j)^t s(j/k) \qquad j = 0,\ldots,k-1 \quad . \tag{F.36}$$

Substituting Equations F.13 and F.14 into Equations F.27 and F.28,

$$\hat{R}(k) = \frac{1}{k+2\lambda_R+m-1}\{\sum_{j=1}^{k}[z(j)-H(j)\hat{x}(j/k)][z(j)-H(j)\hat{x}(j/k)]^t+2R_B\} \tag{F.37}$$

and

$$\hat{Q}(k) = \frac{1}{k+2\lambda_Q+r-1}\sum_{j=1}^{k}\hat{w}(j-1/k)\hat{w}(j-1/k)^t + 2R_Q \quad . \tag{F.38}$$

Comparing Equations F.29, F.32, F.33, F.34, F.36, F.37, and F.38 with their counterparts in Chapter VI (Equations 6.12 through 6.16 and Equations 6.24 and 6.25), one sees that these necessary conditions are the same except for the scale factors in the R and Q estimators. However, for large k, they are essentially the same.

Equation F.26 will now be derived. Let T be defined as

$$T = .5\sum_{p=1}^{k}[z(p) - H(p)x(p)][z(p) - H(p)x(p)]^t + B \quad . \tag{F.39}$$

From Equation F.39,

$$T = .5[z(n) - H(n)x(n)][z(n) - H(n)x(n)]^t + .5\sum_{p\neq n}[z(p) - H(p)x(p)][z(p) - H(p)x(p)]^t + B \quad . \tag{F.40}$$

This is equivalent to considering (to simplify notation)

$$T = .5[z - Hx][z - Hx]^t + B' = F(x) + B' \qquad \text{(F.41)}$$

where B' does not contain the vector x. Then,

$$\frac{\partial \ln|T|}{\partial x} = \frac{\partial \ln|T|}{\partial|T|}\frac{\partial|T|}{\partial x} = \frac{1}{|T|}\frac{\partial|T|}{\partial x} \quad . \qquad \text{(F.42)}$$

Let x_k be the k^{th} element of the vector x. Then

$$\frac{\partial|T|}{\partial x_k} = \sum_{e=1}^{m}\sum_{i=1}^{m}\frac{\partial|T|}{\partial F(x)_{ei}}\frac{\partial F(x)_{ei}}{\partial x_k} \quad . \qquad \text{(F.43)}$$

Now using a relationship developed by Shellenbarger (17),

$$\frac{\partial|T|}{\partial F(x)_{ei}} = \text{cofactor } (T_{ei}) \triangleq C^*_{ei} \quad . \qquad \text{(F.44)}$$

From Equation F.41,

$$F(x)_{ei} = .5\{z_e z_i - \sum_{j=1}^{n} H_{ej}x_j z_i - \sum_{j=1}^{n} H_{ij}x_j z_e + (\sum_{j=1}^{n} H_{ej}x_j)(\sum_{j=1}^{n} H_{ij}x_j)\} \qquad \text{(F.45)}$$

so that

$$\frac{\partial F(x)_{ei}}{\partial x_k} = -.5\{H_{ek}(z_i - \sum_{j=1}^{n} H_{ij}x_j) + H_{ik}(z_e - \sum_{j=1}^{n} H_{ej}x_j)\} \quad . \qquad \text{(F.46)}$$

Substituting Equations F.44 and F.46 into F.43,

$$\frac{\partial |T|}{\partial x_k} = -.5 \sum_{e=1}^{m} \sum_{i=1}^{m} C^*_{ei} \{H_{ek}(z_i - \sum_{j=1}^{n} H_{ij}x_j) + H_{ik}(z_e - \sum_{j=1}^{n} H_{ej}x_j)\}. \quad \text{(F.47)}$$

Since T is symmetric, then $C^*_{ei} = C^*_{ie}$ which results in

$$\frac{\partial |T|}{\partial x_k} = - \sum_{e=1}^{m} \sum_{i=1}^{m} C^*_{ei} H_{ek}(z_i - \sum_{j=1}^{n} H_{ij}x_j) \quad . \quad \text{(F.48)}$$

Equation F.48 can be rewritten as

$$\frac{\partial |T|}{\partial x_k} = - \sum_{e=1}^{m} H_{ek}(\sum_{i=1}^{m} C^*_{ei}(z_i - \sum_{j=1}^{n} H_{ij}x_j))$$

$$= - \sum_{e=1}^{m} H_{ek}(\sum_{i=1}^{m} C^*_{ei}[z - Hx]_i)$$

$$= - \sum_{e=1}^{m} H_{ek}[(\text{adjoint } T)(z - Hx)]_e$$

$$= -H^t[(\text{adjoint } T)(z - Hx)]_k \quad . \quad \text{(F.49)}$$

So that

$$\frac{\partial |T|}{\partial x} = - H^t(\text{adjoint } T)(z - Hx) \quad . \quad \text{(F.50)}$$

From Equations F.50 and F.42,

$$\frac{\partial \ln |T|}{\partial x} = - H^t T^{-1}(z - Hx) \quad . \quad \text{(F.51)}$$

Referring back to the simplified notational change between Equations F.40 and F.41, it can be seen that Equation F.26 is valid.

XVII. APPENDIX G

First, it will be shown that for any given p.d. R and Q, the extremizing solution of Equations 6.12 through 6.16 actually yields the minimum of I** with respect to X(k). Then, for a given X(k), it will be shown that I** is minimized with respect to R and Q by the estimates in Equation 6.17 (or 6.24) and Equation 6.8 (or 6.25) if the estimates are p.d. Assuming that the cost function is convex, then the MAP algorithm will be shown to converge to the overall minimum of the cost function. Finally, a method will be given to determine the "average" properties of the a posteriori density so that some judgment can be made as to the applicability of the MAP criterion for any particular system.

Now it has been shown that the minimum of the cost function I** is equivalent to the maximum of the a posteriori density $P[X(k),Q,R/Z(k)]$ which can be rewritten as

$$P[X(k),Q,R/Z(k)] = P[X(k)/Q,R,Z(k)]P[Q,R/Z(k)] \qquad (G.1)$$

It will be shown later that under the assumptions made in this thesis, $P[X(k)/Q,R,Z(k)]$ is a multivariate Gaussian density with a mean consisting of the fixed interval smoothed estimates of X(k). Since, the Gaussian density is unimodal and symmetric about the mean, the fixed interval smoothed estimates correspond to the only maximum of $P[X(k)/Q,R,Z(k)]$.

Thus, for fixed R and Q, the fixed-interval smoothed state estimates correspond to the only maximum of $P[X(k),Q,R/Z(k)]$ or equivalently, the only minimum of I**. Therefore, for given $\hat{R}(k)$ and $\hat{Q}(k)$ in Equations 6.12 through 6.16, the extremum solutions $\hat{x}(j/k)$, $j = 0,1,\ldots,k$ are the unique minimizing values for I**. This will be true for any positive definite values of $\hat{R}(k)$ and $\hat{Q}(k)$.

A similar statement can be made when X(k) is assumed fixed and the maximum of $P[X(k),Q,R/Z(k)]$ with respect to Q and R is desired. It is shown in Anderson (22) that the minimum of

$$I_{11}(R) = \frac{k}{2} \ln|R| + \frac{1}{2} \sum_{j=1}^{k} \|z(j) - H(j)x(j)\|^2_{R^{-1}} \qquad (G.2)$$

is at

$$R_{min} = \frac{1}{k} \sum_{j=1}^{k} [z(j) - H(j)x(j)][z(j) - H(j)x(j)]^t \qquad (G.3)$$

if

$$\sum_{j=1}^{k} [z(j) - H(j)x(j)][z(j) - H(j)x(j)]^t$$

is p.d.. Also, since G(j-1) was assumed invertible,

$$|G(j-1)QG(j-1)^t| = |G(j-1)|^2|Q| \qquad (G.4)$$

then

$$\frac{1}{2} \sum_{j=1}^{k} \ln|G(j-1)QG(j-1)^t| = \frac{k}{2} \ln|Q| + \sum_{j=1}^{k} \ln|G(j-1)| . \qquad (G.5)$$

Therefore, the minimum of

$$I_{22}(Q) = \frac{1}{2} \sum_{j=1}^{k} \ln|G(j-1)QG(j-1)| + \frac{1}{2} \sum_{j=1}^{k} ||w(j-1)||^2 Q^{-1} \tag{G.6}$$

is at (23)

$$Q_{min} = \frac{1}{k} \sum_{j=1}^{k} w(j-1)w(j-1)^t \tag{G.7}$$

if

$$\sum_{j=1}^{k} w(j-1)w(j-1)^t$$

is positive definite. Consequently, for given X(k), the cost function I** has a minimum at the values indicated in Equations G.3 and G.7 if these values are p.d. Note that these are precisely the necessary conditions of Equations 6.17 and 6.18. Similar statements can also be made with respect to Equations 6.24 and 6.25 being the minimizing values of I* in Equation 6.22 when X(k) is given.

If the cost function is convex, then the gradient vector will equal the null vector only at the global minimum (30). Thus, at any other point, at least some of the components of the gradient in the X(k) and Q, R subspaces will be nonzero. Consequently, if the estimates of Q and R are positive definite, the MAP algorithm successively will seek

the minimum of the cost function within each subspace, given a fixed vector from the other subspace. It will continue to do this until successive attempts to minimize yield the same point. Since the gradient is the null vector only at the global minimum, then only at this point will the gradient components in the X(k) and Q, R subspaces be all zero. Thus, only at this point will successive minimizations of the MAP algorithm yield the same point. Essentially, the convexity assumption rules out such things as saddle points, where the MAP algorithm may get "hung up."

Now it will be shown that $P[X(k)/Q,R,Z(k)]$ is a multivariate Gaussian density with a mean consisting of the fixed interval smoothed estimates of X(k). From Equation 2.2,

$$\begin{aligned} z(1) &= H(1)x(1) + v(1) \\ &\vdots \\ z(k) &= H(k)x(k) + v(k) \end{aligned} \tag{G.8}$$

or

$$Z(k) = F(k)X(k) + V(k) \quad . \tag{G.9}$$

The Bayesian equivalent to $P[X(k)/,Q,R,Z(k)]$ is

$$P[X(k)/Q,R,Z(k)] = \frac{P[Z(k)/Q,R,X(k)]P[X(k)/,Q,R]}{P[Z(k)/,Q,R]} \quad . \tag{G.10}$$

It can be shown that since w(j), v(j), and x(0) are all Gaussian, then

$$(Z(k)/Q,R,X(k)) \sim N[F(k)X(k), C_{V(k)}] \tag{G.11}$$

$$(X(k)/Q,R,) \sim N[0, C_{X(k)}] \tag{G.12}$$

$$(Z(k)/Q,R) \sim N[0, C_{Z(k)}] \tag{G.13}$$

where

$$C_{V(k)} \triangleq E[V(k)V(k)^t] = \begin{bmatrix} R & & & \\ & \ddots & & \\ & & R & 0 \\ & & & \ddots \\ & & & R \end{bmatrix} \tag{G.14}$$

$$C_{X(k)} \triangleq E[X(k)X(k)^t] \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & & \\ \vdots & & \ddots & \\ M_{k1} & & & M_{kk} \end{bmatrix} \tag{G.15}$$

where

$$M_{ij} \triangleq E[X(i)X(j)^t] \tag{G.16}$$

and

$$C_{Z(k)} \triangleq F(k)C_{X(k)}F(k)^t + C_{V(k)} \quad . \tag{G.17}$$

Using Equations G.11 through G.17 in Equation G.10, it can be shown (31) that

$$P[X(k)/Q,R,Z(k)] = \frac{\exp\frac{1}{2}\{\|X(k)-C_{X(k)}F(k)^{t}C_{Z(k)}^{-1}Z(k)\|^{2}C_{X(k)}^{*-1}\}}{(2\pi)^{\frac{kn}{2}}|C_{X(k)}^{*}|^{\frac{1}{2}}} \tag{G.18}$$

where

$$C_{X(k)}^{*-1} = C_{X(k)}^{-1} + F(k)^{t}C_{Z(k)}^{-1}F(k) \quad . \tag{G.19}$$

With much work, it can be shown that $C_{X(k)}F(k)^{t}C_{Z(k)}^{-1}Z(k)$ is the vector of fixed interval smoothed estimates of $X(k)$.

A method will now be given to enable an assessment of the MAP criterion without having to actually run Monte Carlo simulations. This will indicate how the a posteriori density behaves "on the average" for a particular system. Now from Equation G.1,

$$\max_{X(k)} P[X(k),Q,R,/Z(k)] = \max_{X(k)} P[X(k)/Q,R,Z(k)]P[Q,R/Z(k)]. \tag{G.20}$$

Now the maximum of $P[X(k)/Q,R,Z(k)]$ with respect to $X(k)$ is from Equations G.18 and G.19

$$\max_{X(k)} P[X(k)/Q,R,Z(k)] = \frac{1}{(2\pi)^{\frac{kn}{2}}|C_{X(k)}^{*}|^{\frac{1}{2}}} \tag{G.21}$$

or from Equation G.19,

$$\max_{X(k)} P[X(k)/Q,R,Z(k)] = \frac{|C_{Z(k)}|^{\frac{1}{2}}}{(2\pi)^{\frac{kn}{2}}|C_{X(k)}|^{\frac{1}{2}}|C_{V(k)}|^{\frac{1}{2}}} \quad . \tag{G.22}$$

From Equation G.14,

$$|C_{V(k)}| = k|R| \quad . \tag{G.23}$$

To evaluate $|C_{X(k)}|$, just look at M_{ij} of Equation G.16. From Equation 2.1, for $i < j$,

$$x(j) = A(j,i)x(i) + f(w(i),w(i+1,),\ldots,w(j-1)) \quad . \tag{G.24}$$

Therefore,

$$M_{ij} = E[x(i)x(j)^t] = M_{ii}A(j,i)^t \quad . \tag{G.25}$$

Similarly, for $i > j$,

$$M_{ij} = A(j,i)M_{ii} \quad . \tag{G.26}$$

Also, from Equation 2.1,

$$M_{ii} = A(i,i-1)M_{i-1\ i-1}A(i,i-1)^t + G(i-1)QG(i-1)^t. \tag{G.27}$$

Now since $C_{X(1)} = M_{11}$, then $|C_{X(1)}| = |M_{11}|$. By Equations G.15, G.25 and G.26,

$$C_{X(2)} = \left[\begin{array}{c|c} M_{11} & M_{11}A(2,1)^t \\ \hline A(2,1)M_{11} & M_{22} \end{array}\right]$$

so that

$$|C_{X(2)}| = |M_{11}||M_{22} - A(2,1)M_{11}M_{11}^{-1}M_{11}A(2,1)^t|$$

$$= |M_{11}||G(1)QG(1)^t| \quad . \tag{G.28}$$

Similarly,

$$|C_{X(3)}| = |M_{11}||G(1)QG(1)^t||G(2)QG(2)^t| \qquad (G.29)$$

continuing on by induction, it can be shown that

$$|C_{X(k)}| = |M_{11}| \prod_{j=1}^{k-1} |G(j)QG(j)^t| \qquad (G.30)$$

or if G(j) is stationary, then

$$|C_{X(k)}| = |M_{11}||GQG^t|^{k-1} \quad . \qquad (G.31)$$

From Equations G.8, G.9, G.17, G.14, and G.15,

$$C_{Z(k)} = \begin{bmatrix} H(1)M_{11}H(1)^t + R & H(1)M_{12}H(2)^t & \cdots & H(1)M_{1k}H(k)^t \\ H(2)M_{21}H(1)^t & H(2)M_{22}H(2) + R & \cdots & \\ \vdots & & & \vdots \\ H(k)M_{k1}H(1)^t & \cdots & & H(k)M_{kk}H(k) + R \end{bmatrix} \qquad (G.32)$$

It appears that there is no simple way of calculating $|C_{Z(k)}|$ other than by some "brute force" method.

Thus, for any particular system, the determinants in Equation G.22 can be calculated for any given R and Q.

Also,

$$P[Q,R/Z(k)] = \frac{P[Z(k)/Q,R]P[Q,R]}{P[Z(k)]} \quad . \qquad (G.33)$$

Substituting Equations G.33, G.22 into G.20, then

$$L = -\frac{1}{k}\ln\{\max_{X(k)} P[X(k),Q,R/Z(k)]\}$$

$$= \frac{n}{2}\ln(2\pi) + \frac{1}{2k}\ln|C_{X(k)}| + \frac{1}{2k}\ln|C_{V(k)}| - \frac{1}{2k}\ln|C_{Z(k)}|$$

$$+ \frac{1}{k}\ln P[Z(k)] - \frac{1}{k}\ln P[Z(k)/Q,R] - \frac{1}{k}\ln P[Q,R]. \quad (G.34)$$

Since $-\frac{1}{k}\ln(\ )$ is a monotonically decreasing function, then the values of Q and R that maximize the quantity in Equation G.20 are the same as the minimizing values for L in Equation G.34.

Kashyap (13) has pointed out that for a stationary system and for large k,

$$-\frac{1}{k}\ln P[Z(k)/Q,R]$$

approaches

$$E\{-\ln P[z(i)|z(i-1),R,Q]|Q_t,R_t\}$$

which yields,

$$-\frac{1}{k}\ln P[Z(k)|Q,R] \tilde{=} + \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln|Y_{ss}| + Tr[Y_t Y_{ss}^{-1}] \quad (G.35)$$

where

Y_t = steady state value of $H(i)P(i/k-1)H(i)^t + R_t$ from a Kalman-filter set to true values of R and Q.

Y_{ss} = steady state value of $H(i)P(i/i-1)H(i)^t + R$ from a Kalman-filter set to the given values of R and Q.

Thus, L becomes (neglecting terms that are independent of Q and R), from Equations G.31 and G.23

$$L^* = \frac{1}{2} \ln|GQG^t| + \frac{1}{2k} \ln\left\{\frac{|M_{11}|}{|GQG^t|}\right\} + \frac{1}{2} \ln|R|$$

$$+ \frac{1}{2} \ln|Y_{ss}| + Tr[Y_t Y_{ss}^{-1}] - \frac{1}{k} \ln P[Q,R]$$

$$- \frac{1}{2k} \ln|C_{Z(k)}| \quad . \tag{G.36}$$

For given values of R and Q, L*, can be calculated without resorting to a Monte Carlo simulation. This "average" cost function can then be examined in the subspace of the R and Q variables by generating L* for an assumed true R and Q, given many values of R and Q. This will then give an "average" indication as to whether the MAP criterion is applicable.

XVIII. APPENDIX H

The mean and dispersion of the R estimator of Equation 6.18 will be determined in this appendix. From Equation 6.18,

$$\hat{R}(k) = \frac{1}{k} \sum_{j=1}^{k} \tilde{z}(j/k)\tilde{z}(j/k) \tag{H.1}$$

where

$$\tilde{z}(j/k) = z(j) - H(j)\hat{x}(j/k) \quad . \tag{H.2}$$

Galles (32) shows that

$$\hat{x}(j/k) = \hat{x}(j/j) + \sum_{i=j+1}^{k} K(j,i)\tilde{z}(i/i-1) \tag{H.3}$$

where $K(j,i)$ is the fixed-point smoothing gain that relates $\hat{x}(j/i)$ to $\hat{x}(j/i-1)$. Substituting Equation 2.9 into H.3 the result into Equation H.2, and recognizing the definition for $\tilde{z}(j/j-1)$,

$$\tilde{z}(j/k) = [I - H(j)K(j)]\tilde{z}(j/j-1) - H(j) \sum_{i=j+1}^{k} K(j,i)\tilde{z}(i/i-1). \tag{H.4}$$

Now, defining

$$Y_a(e,j) \triangleq E[\tilde{z}(e/e-1)\tilde{z}(j/j-1)^t] \tag{H.5}$$

then from Equation H.4,

$$E[\tilde{z}(e/k)\tilde{z}(j/k)^t] = [I - H(e)K(e)]Y_a(e,j)[I - H(j)K(j)]^t$$

$$- [I - H(e)K(e)]\{\sum_{i=j+1}^{k} Y_a(e,i)K(j,i)^t\}H(j)^t$$

$$- H(e)\{\sum_{i=e+1}^{k} K(e,i)Y_a(i,j)\}[I - H(j)K(j)]^t$$

$$+ H(e)\{\sum_{p=j+1}^{k}\sum_{i=j+1}^{k} K(e,p)Y_a(p,i)K(j,i)^t\}H(j)^t \quad . \tag{H.6}$$

To simplify the equations, it will be assumed that "good" estimates of R and Q are used in the smoothing equations so that they can essentially be considered as the true values. Then from Equation 4.20, the above assumption results in

$$Y_a(i,j) = \begin{cases} H(i)P(i/i-1)H(i)^t + R_t & i=j \\ 0 & i\neq j \end{cases} \tag{H.7}$$

since in this case, $P_a(j/j-1) = P_c(j/j-1) = P(j/j-1)$. Thus, Equation H.6 simplifies to

$$E[\tilde{z}(j/k)\tilde{z}(j/k)] = [I - H(j)K(j)]Y_a(j,j)[I - H(j)K(j)]^t$$

$$+ H(j)\{\sum_{i=j+1}^{k} K(j,i)Y_a(i,i)K(j,i)^t\}H(j)^t \quad . \tag{H.8}$$

From Galles (32), the smoothing covariance is

$$P(j/k) = P(j/k-1) - K(j,k)Y_a(k,k)K(j,k)^t \quad . \tag{H.9}$$

By successively substituting Equation H.9 into itself, then

$$P(j/k) = P(j/j) - \sum_{i=j+1}^{k} K(j,i)Y_a(i,i)K(j,i)^t \tag{H.10}$$

also, from Equation 2.12,

$$K(j) = P(j/j-1)H(j)^t Y_a(j,j)^{-1} \quad . \tag{H.11}$$

Substituting Equations H.11 and H.10 into H.8, and using Equation 2.14,

$$E[\tilde{z}(j/k)\tilde{z}(j/k)] = R_t - H(j)P(j/k)H(j)^t \tag{H.12}$$

so that from Equation H.1,

$$E[\hat{R}(k)] = R_t - \frac{1}{k}\sum_{j=1}^{k} H(j)P(j/k)H(j)^t \quad . \tag{H.13}$$

Hence, in general, it can be seen that $\hat{R}(k)$ is biased even when using the optimal smoothing estimates.

By a procedure, very similar to the one used in Appendices A and B, it can be shown that the dispersion of $\hat{R}(k)$ about its mean value goes to zero as k gets large under the same observability and controllability assumptions used in Appendix B. In general, then, the successive

reiterations will never converge to the true value of R.

The analogous expressions for the Q estimator can be similarly derived.